Submitted to Mathematische Nachrichten, 2006
Registered as Technical Report 2006-1 at the University of Siena.

A continuation principle for a class of periodically perturbed autonomous systems

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Key words autonomous systems, periodic perturbations, limit cycles, topological degree, periodic solutions.
MSC (2000) 34C25, 34A34, 34D10, 47H11, 47H14

The paper deals with a T-periodically perturbed autonomous system in $\mathbb{R}^n$ of the form

$$\dot{x} = \psi(x) + \varepsilon\phi(t, x, \varepsilon) \tag{PS}$$

with $\varepsilon > 0$ small. The main goal of the paper is to provide conditions ensuring the existence of T-periodic solutions to (PS) belonging to a given open set $W \subset C([0, T], \mathbb{R}^n)$. This problem is considered in the case when the boundary ∂W of W contains at most a finite number of nondegenerate T-periodic solutions of the autonomous system $\dot{x} = \psi(x)$. The starting point of our approach is the following property due to Malkin: if for any T-periodic limit cycle x_0 of $\dot{x} = \psi(x)$ belonging to ∂W the so-called bifurcation function $f_{x_0}(\theta)$, $\theta \in [0, T]$, associated to x_0, see (1.11), satisfies the condition $f_{x_0}(0) \neq 0$ then the integral operator

$$(Q_\varepsilon x)(t) = x(T) + \int_0^t \psi(x(\tau))d\tau + \varepsilon \int_0^t \phi(\tau, x(\tau), \varepsilon)d\tau, \quad t \in [0, T],$$

does not have fixed points on ∂W for all $\varepsilon > 0$ sufficiently small. By means of the Malkin's bifurcation function we then establish a formula to evaluate the Leray-Schauder topological degree of $I - Q_\varepsilon$ on W. This formula permits to state existence results that generalize or improve several results of the existing literature. In particular, we extend a continuation principle due to Capietto, Mawhin and Zanolin where it is assumed that ∂W does not contain any T-periodic solutions of the unperturbed system. Moreover, we obtain generalizations or improvements of some existence results due to Malkin and Loud.

1 Introduction

The aim of this paper is to provide conditions ensuring the existence of T-periodic solutions to the T-periodically perturbed system of the form

$$\dot{x} = \psi(x) + \varepsilon\phi(t, x, \varepsilon) \tag{1.1}$$

belonging to a given set $W \subset C([0, T], \mathbb{R}^n)$. Here we assume that

$$\psi \in C^1(\mathbb{R}^n, \mathbb{R}^n) \text{ and } \phi : \mathbb{R} \times \mathbb{R}^n \times [0, 1] \to \mathbb{R}^n \text{ satisfies Carathéodory type conditions,} \tag{1.2}$$

Supported by the research project GNAMPA :"Qualitative Analysis and Control of Hybrid Systems", by RFBR grants 02-01-00189, 05-01-00100, by U.S.CRDF - RF Ministry of Education grant VZ-010 and by President of Russian Federation Fellowship for Scientific Training Abroad.

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i.e. $\phi(\cdot,x,\varepsilon)$ is (Lebesgue) measurable for each (x,ε), $\phi(t,\cdot,\cdot)$ is continuous for almost all (a.a.) t and, for each $r>0$ there exists $b_r \in L^1([0,T],\mathbb{R}_+)$ such that $|\phi(t,x,\varepsilon)| \le b_r(t)$ for a.a. $t\in[0,T]$ and all $|x|\le r$, $\varepsilon\in[0,1]$. Moreover, ϕ is T-periodic in time and any solution $x\in C([0,T],\mathbb{R}^n)$ to (1.1) satisfying the boundary condition

$$x(0)=x(T) \tag{1.3}$$

will be called a T-periodic solution to (1.1). Any T-periodic function $x\in C([0,T],\mathbb{R}^n)$ is considered as extended from $[0,T]$ to $\mathbb{R}$ by T-periodicity. Moreover, any constant function $x\in C([0,T],\mathbb{R}^n)$ is identified with the vector $x(0)$ of $\mathbb{R}^n$. Let $Q_\varepsilon : C([0,T],\mathbb{R}^n)\to C([0,T],\mathbb{R}^n)$ be the integral operator given by

$$(Q_\varepsilon x)(t) = x(T) + \int_0^t \psi(x(\tau))d\tau + \varepsilon\int_0^t \phi(\tau,x(\tau),\varepsilon)d\tau, \quad t\in[0,T],\ \varepsilon>0,$$

whose fixed points are T-periodic solutions to (1.1). In the case when

$$Q_0x\neq x \quad \text{for any } x\in\partial W \tag{1.4}$$

and

$$d_{\mathbb{R}^n}(\psi, W\cap\mathbb{R}^n)\neq 0, \tag{1.5}$$

where $d_{\mathbb{R}^n}(\psi, W\cap\mathbb{R}^n)$ is the Brouwer topological degree of ψ in $W\cap\mathbb{R}^n$, the existence problem of T-periodic solutions to (1.1) has been solved by Capietto, Mawhin and Zanolin in [3]. In fact, they proved ([3], Corollary 1), that under conditions (1.4) and (1.5) the following formula holds

$$d(I-Q_0,W) = (-1)^n d_{\mathbb{R}^n}(\psi, W\cap\mathbb{R}^n), \tag{1.6}$$

where $d(I-Q_0,W)$ is the Leray-Schauder topological degree of $I-Q_0$ in W. It follows from (1.6) that

$$d(I-Q_\varepsilon,W) = (-1)^n d_{\mathbb{R}^n}(\psi, W\cap\mathbb{R}^n) \tag{1.7}$$

for any $\varepsilon>0$ sufficiently small. Therefore under conditions (1.4) and (1.5) system (1.1) has a T-periodic solution in W for any perturbation term ϕ and any sufficiently small $\varepsilon>0$. Observe that the assumption (1.5) implies that the set W contains a constant solution of

$$\dot{x}=\psi(x). \tag{1.8}$$

In [3] many relevant examples satisfying conditions (1.4) and (1.5) are provided. Moreover, the authors have focused several results due to I. Berstein and A. Halanay, J. Cronin, A. Lando, E. Muhamadiev and others, which have been generalized or improved.

The main goal of this paper is to provide conditions on the perturbation term ϕ in such a way that, for $\varepsilon>0$ sufficiently small, $d(I-Q_\varepsilon,W)$ is defined and different from zero for a wider class of sets W. Indeed, through the paper we will not require (1.4), i.e. we will allow ∂W to contain fixed point of Q_0. Under this more general condition, we will establish a formula for $d(I-Q_\varepsilon,W)$ that guarantees, under suitable conditions on ϕ, that $d(I-Q_\varepsilon,W)\neq 0$ even in the case when $d_{\mathbb{R}^n}(\psi,W\cap\mathbb{R}^n)=0$. Precisely, we assume that

$$\text{the set } \mathfrak{S}_W = \{x\in\partial W : Q_0x = x\} \text{ is finite,} \tag{1.9}$$

and for any $x_0\in\mathfrak{S}_W$ the linearized system

$$\dot{y} = \psi'(x_0(t))y \tag{1.10}$$

has the characteristic multiplier 1 of multiplicity 1, i. e. any $x_0\in\mathfrak{S}_W$ is a nondegenerate limit cycle of (1.8). It is clear that, under assumption (1.9), the topological degree $d(I-Q_0,W)$ is not necessarily defined. The approach proposed in this paper to overcome this difficulty consists in introducing the Malkin's bifurcation function

$$f_{x_0}(\theta) = \mathrm{sign}\,\langle \dot{x}_0(0), z_0(0)\rangle \int\limits_0^T \langle z_0(\tau), \phi(\tau-\theta, x_0(\tau),0)\rangle\, d\tau, \tag{1.11}$$

where z_0 is a nontrivial T-periodic solution of the adjoint system

$$\dot{z} = -(\psi'(x_0(t)))^* z. \tag{1.12}$$

From [13] (or [12], Theorem p. 387) we have that if

$$f_x(0) \neq 0 \quad \text{for any } x \in \mathfrak{S}_W \tag{1.13}$$

then

$$\text{for every } \varepsilon > 0 \text{ sufficiently small the topological degree } d(I - Q_\varepsilon, W) \text{ is defined.} \tag{1.14}$$

In this paper we prove in Theorem 2.4 that if (1.13) is satisfied then, for all $\varepsilon > 0$ sufficiently small, we have

$$d(I - Q_\varepsilon, W) = (-1)^n d_{\mathbb{R}^n}(\psi, W \cap \mathbb{R}^n) - \sum_{x \in \mathfrak{S}_W:\ \Theta_W(x) \neq \emptyset} (-1)^{\beta(x)} d_{\mathbb{R}}\left(f_x, (0, \min\{\Theta_W(x)\})\right), \tag{1.15}$$

where

$$\Theta_W(x) = \{\theta_0 \in (0,T) : S_{\theta_0} x \in \partial W,\ S_\theta x \in W \text{ for any } \theta \in (0, \theta_0)\}, \quad \text{for any } x \in \mathfrak{S}_W,$$

$$(S_\theta x)(t) = x(t + \theta) \quad \text{and}$$

$\beta(x_0)$ is the sum of the multiplicities of the characteristic multipliers greater than 1 of (1.10).

Therefore it follows that for any perturbation term ϕ satisfying conditions (1.9), (1.13) if

$$(-1)^n d_{\mathbb{R}^n}(\psi, W \cap \mathbb{R}^n) - \sum_{x \in \mathfrak{S}_W:\ \Theta_W(x) \neq \emptyset} (-1)^{\beta(x)} d_{\mathbb{R}}\left(f_x, (0, \min\{\Theta_W(x)\})\right) \neq 0 \tag{1.16}$$

then, for any $\varepsilon > 0$ sufficiently small, system (1.1) has a T-periodic solution in W. Observe that if (1.5) is not satisfied, but there exist at least one $x \in \mathfrak{S}_W$ such that $\Theta_W(x) \neq \emptyset$ then assumption (1.16) can be fulfilled by a suitable choice of the perturbation term ϕ. In this sense assumption (1.16) is weaker than (1.5).

The second term on the right hand side of (1.15) is similar to that of the Krasnosel'skii-Zabreyko's formula for computing the index of a degenerate fixed point of Q_0 by means of a reduction to a subspace (in our case one-dimensional), see ([8], formula 24.13). However, the related Krasnosel'skii-Zabreyko result ([8], Theorem 24.1) can be applied only in the case when the operator Q_0 has a particular form ensuring that Q_0 has only isolated fixed points. This is not our case since any T-periodic cycle of (1.8) is a non-isolated fixed point of Q_0.

Furthermore, observe that the case when $\mathfrak{S}_W$ is nonempty was already treated in the literature. For instance, if $\psi = 0$ then any solution of (1.8) is T-periodic, $\mathfrak{S}_W = \partial W$ and $d(I - Q_\varepsilon, W)$ can be evaluated by means of the following formula due to Mawhin, see ([14] and [15])

$$d(I - Q_\varepsilon, W) = d_{\mathbb{R}^n}\left(-\int_0^T \phi(\tau, \cdot, 0) d\tau, W \cap \mathbb{R}^n\right). \tag{1.17}$$

Mawhin proved (1.17) in the case when $\varepsilon > 0$ is not necessarily small. The same formula can be also used when $\psi \neq 0$, but any solution of (1.8) in $\overline{W}$ is T-periodic (see [20], formulas 3.1-3.3). This assumption has been considerably weakened by the authors in [6] for a wide class of sets W. Specifically, in [6] it was assumed that there exists $U \subset \mathbb{R}^n$ such that W is the set of all continuous functions from $[0,T]$ to U and any point of ∂U is the initial condition of a T-periodic solution to (1.8), namely it was still assumed that $\mathfrak{S}_W$ is an infinite subset of ∂W. For $\varepsilon > 0$ sufficiently small formula (1.17) was expressed as follows, see also ([5], formula 46),

$$d(I - Q_\varepsilon, W) = d_{\mathbb{R}^n}\left(-\int_0^T \left(x'_{(2)}(\tau, \cdot)\right)^{-1} \phi(\tau, x(\tau, \cdot), 0) d\tau, U\right), \tag{1.18}$$

where $x(\cdot,\xi)$ is the solution of (1.8) satisfying $x(0,\xi) = \xi$. Hence, if $d_{\mathbb{R}^n}(\psi, W \cap \mathbb{R}^n) = 0$ then (1.15) can be considered as a further development of (1.17) for the special case when $\mathfrak{S}_W$ is finite. In fact, the following formula holds, (see (2.33) in the proof of next Theorem 2.1),

$$f_{x_0}(\theta) = \text{sign}\,\langle \dot{x}_0(0), z_0(0)\rangle \left\langle \int_0^T \left(x'_{(2)}(\tau,\cdot)\right)^{-1} \phi(\tau, x(\tau,\cdot), 0)d\tau, z_0(\theta)\right\rangle \quad \text{for any } \theta \in [0,T].$$

The paper is organized as follows. Section 2 is devoted to the proof of formula (1.15) and its variants. In section 3 by different choices of the set W we obtain several new existence results for T-periodic solutions to (1.1). In particular, we generalize or improve some existence results due to Loud and Malkin proved in [11] and [13] respectively.

2 Main results

Let $x^{-1}(t,\cdot)$ be the inverse of $x(t,\cdot)$, that is $x(t, x^{-1}(t,\xi)) = \xi$ for any $t \in \mathbb{R}$ and any $\xi \in \mathbb{R}^n$. For any set V of $\mathbb{R}^n$, define the set W_V of $C([0,T],\mathbb{R}^n)$ by

$$W_V = \left\{\widehat{x} \in C([0,T],\mathbb{R}^n) : x^{-1}(t,\widehat{x}(t)) \in V, \text{ for any } t \in [0,T]\right\}.$$

Clearly, W_V is open in $C([0,T],\mathbb{R}^n)$ provided that V is open in $\mathbb{R}^n$. In the sequel by $B_\delta(A)$ we denote the δ-neighborhood of the set A with respect to the norm of the space containing A. The following result is crucial for the proof of our Theorem 2.4, but it has also an independent interest for some applications as shown in Section 3.

Theorem 2.1 *Let x_0 be a nondegenerate T-periodic limit cycle of system (1.8). Let $0 \le \theta_1 < \theta_2 \le \theta_1 + \frac{T}{p}$, where $p \in \mathbb{N}$ and $\frac{T}{p}$ is the least period of x_0. Assume that $f_{x_0}(\theta_1) \neq 0$ and $f_{x_0}(\theta_2) \neq 0$. Then, for a given $\alpha > 0$, there exist $\delta_0 > 0$ and a family of open sets $\{V_\delta\}_{\delta\in(0,\delta_0]}$ satisfying the properties*
1) $x_0((\theta_1,\theta_2)) \subset V_\delta \subset B_\delta(x_0((\theta_1,\theta_2)))$,
2) $\partial V_\delta \cap x_0([\theta_1,\theta_2]) = \{x_0(\theta_1), x_0(\theta_2)\}$
and such that for any $\delta \in (0,\delta_0]$ and any $\varepsilon \in (0,\delta^{1+\alpha}]$ the degree $d(I - Q_\varepsilon, W_{V_\delta})$ is defined and it can be evaluated by the following formula

$$d(I - Q_\varepsilon, W_{V_\delta}) = -(-1)^{\beta(x_0)} d_{\mathbb{R}}(f_{x_0}, (\theta_1,\theta_2)).$$

Now we introduce some preliminary notions and results necessary for the proof of the theorem. Let x_0 be a nondegenerate limit cycle of (1.8), then there exists, see e.g. ([4], Lemma 1 Chap. IV, §20), a fundamental matrix $Y(t)$ of system (1.10) having the form

$$Y(t) = \Phi(t)\begin{pmatrix} e^{\Lambda t} & 0_{n-1\times 1} \\ 0_{1\times n-1} & 1 \end{pmatrix}, \tag{2.1}$$

where Φ is a T-periodic Floquet matrix and Λ is a constant $(n-1)\times(n-1)$-matrix with eigenvalues different from 0. In (2.1) it is denoted by $0_{i\times j}$ the $i \times j$ zero matrix, in the sequel we will omit these subindexes when confusion will not arise. For any $\delta > 0$ define the set $C_\delta \subset \mathbb{R}^n$ as follows

$$C_\delta = \left\{\zeta \in \mathbb{R}^n : \|P_{n-1}\zeta\| < \delta,\ \zeta^n \in \left(-\frac{\theta_2-\theta_1}{2}, \frac{\theta_2-\theta_1}{2}\right)\right\},$$

where

$$P_{n-1}\zeta = \begin{pmatrix} \zeta^1 \\ \vdots \\ \zeta^{n-1} \\ 0 \end{pmatrix}$$

ζ^k is the k-th component of the vector ζ and θ_1, θ_2 are as in Theorem 2.1. Let $\Gamma : B_\Delta(C_\delta) \to \Gamma(B_\Delta(C_\delta))$, $\Delta > 0$, be as follows

$$\Gamma(\zeta) = \frac{Y(\zeta^n + \overline{\theta})}{\|Y\|_{M_T}} P_{n-1}\zeta + x_0(\zeta^n + \overline{\theta}),$$

where

$$\overline{\theta} = \frac{\theta_1 + \theta_2}{2} \quad \text{and} \quad \|Y\|_{M_T} = \max_{\theta\in[0,T]} \|Y(\theta)\|.$$

We have the following preliminary properties.

Lemma 2.2 *$\langle Y(\theta)P_{n-1}\zeta, z_0(\theta)\rangle = 0$ for any $\theta \in [0,T]$ and any $\zeta \in \mathbb{R}^n$. Moreover, if $\langle \xi, z_0(\theta)\rangle = 0$ for any $\theta \in [0,T]$, then there exists $\zeta \in \mathbb{R}^n$ such that $\langle Y(\theta)P_{n-1}\zeta, z_0(\theta)\rangle = 0$ for any $\theta \in [0,T]$.*

Proof. Let $\zeta \in \mathbb{R}^n$ and define

$$\widehat{\zeta} = \begin{pmatrix} \left(I - \mathrm{e}^{\Lambda T}\right)^{-1} & 0 \\ 0 & 0 \end{pmatrix} \zeta.$$

By Perron's lemma [18] we have

$$\left\langle Y(\theta+T)P_{n-1}\widehat{\zeta}, z_0(\theta)\right\rangle = \left\langle Y(\theta)P_{n-1}\widehat{\zeta}, z_0(\theta)\right\rangle \quad \text{for any } \theta \in [0,T].$$

Therefore

$$\begin{aligned} 0 &= \left\langle \left(Y(\theta) - Y(\theta+T)\right) P_{n-1}\widehat{\zeta}, z_0(\theta)\right\rangle \\ &= \left\langle \Phi(\theta) \begin{pmatrix} \mathrm{e}^{\Lambda\theta}\left(I - \mathrm{e}^{\Lambda T}\right) & 0 \\ 0 & 0 \end{pmatrix} P_{n-1}\widehat{\zeta}, z_0(\theta)\right\rangle \\ &= \left\langle \Phi(\theta) \begin{pmatrix} \mathrm{e}^{\Lambda\theta} & 0 \\ 0 & 0 \end{pmatrix} P_{n-1}\zeta, z_0(\theta)\right\rangle = \langle Y(\theta)P_{n-1}\zeta, z_0(\theta)\rangle \quad \text{for any } \theta \in [0,T]. \end{aligned}$$

To prove the second assertion define

$$L_\xi = \{\xi \in \mathbb{R}^n : \langle \xi, z_0(\theta)\rangle = 0\}, \quad L_\zeta = \bigcup_{\zeta\in\mathbb{R}^n} Y(\theta)P_{n-1}\zeta.$$

L_ξ and L_ζ are linear subspaces of $\mathbb{R}^n$ and $\dim L_\xi = n-1$. Since, for any $\theta \in [0,T]$, $Y(\theta)P_{n-1}$ is a linear nonsingular map acting from $P_{n-1}\mathbb{R}^n$ to $Y(\theta)P_{n-1}\mathbb{R}^n$, then $\dim L_\zeta = \dim P_{n-1}\mathbb{R}^n = n-1$. But by the first assertion of the lemma $L_\xi \supset L_\zeta$ and thus we can conclude that $L_\xi = L_\zeta$.

□

Lemma 2.3 *For any $\Delta \in (0, \Delta_0]$ and any $\delta \in (0, \delta_0]$ we have that Γ is a homeomorphism of $B_\Delta(C_\delta)$ onto $\Gamma(B_\Delta(C_\delta))$ provided that $\Delta_0 > 0$ and $\delta_0 > 0$ are sufficiently small. Moreover, the set $\Gamma(B_\Delta(C_\delta))$ is open in $\mathbb{R}^n$ and Γ^{-1} is continuously differentiable in $\Gamma(B_\Delta(C_\delta))$.*

Proof. Obviously Γ is continuous. Let us show that $\Gamma : B_\Delta(C_\delta) \to \Gamma(B_\Delta(C_\delta))$ is injective for $\Delta > 0$ and $\delta > 0$ sufficiently small. For this assume the contrary, thus there exist $\{a_k\}_{k\in\mathbb{N}}, \{b_k\}_{k\in\mathbb{N}} \subset \mathbb{R}^n$, $a_k \neq b_k$, $a_k \to a_0$, $b_k \to b_0$ as $k \to \infty$,

$$P_{n-1}a_0 = P_{n-1}b_0 = 0, \tag{2.2}$$

such that

$$\frac{Y(a_k^n)}{\|Y\|_{M_T}} P_{n-1}a_k + x_0(a_k^n) = \frac{Y(b_k^n)}{\|Y\|_{M_T}} P_{n-1}b_k + x_0(b_k^n). \tag{2.3}$$

Without loss of generality we may assume that either $a_k^n = b_k^n$ for any $k \in \mathbb{N}$ or $a_k^n \neq b_k^n$ for any $k \in \mathbb{N}$. Assume that $a_k^n = b_k^n$ for any $k \in \mathbb{N}$, thus

$$Y(a_k^n)(P_{n-1}a_k - P_{n-1}b_k) = 0 \quad \text{for any } k \in \mathbb{N},$$

and so

$$P_{n-1}a_k = P_{n-1}b_k \quad \text{for any } k \in \mathbb{N},$$

contradicting the property that $a_k \neq b_k$ for any $k \in \mathbb{N}$. Consider now the case when $a_k^n \neq b_k^n$ for any $k \in \mathbb{N}$, from (2.3) we have $x_0(a_0^n) = x_0(b_0^n)$. Moreover, since $0 \leq \theta_1 < \theta_2 \leq \theta_1 + \frac{T}{p}$, by our choice of θ_1 and θ_2, for $\Delta > 0$ and $\delta > 0$ sufficiently small we have that $|a_0^n - b_0^n| < \frac{T}{p}$, where $\frac{T}{p}$ is the least period of x_0, thus $a_0^n = b_0^n =: \theta_0$. By using Lemma 2.2, from (2.3) we have

$$\langle x_0(a_k^n) - x_0(b_k^n), z_0(a_k^n)\rangle = \left\langle \frac{Y(b_k^n)}{\|Y\|_{M_T}} P_{n-1}b_k^n, z_0(a_k^n) \right\rangle = \left\langle \frac{Y(b_k^n) - Y(a_k^n)}{\|Y\|_{M_T}} P_{n-1}b_k^n, z_0(a_k^n) \right\rangle,$$

or equivalently, by dividing by $a_k^n - b_k^n$

$$\left\langle \frac{x_0(a_k^n) - x_0(b_k^n)}{a_k^n - b_k^n}, z_0(a_k^n) \right\rangle = -\frac{1}{\|Y\|_{M_T}} \left\langle \frac{Y(a_k^n) - Y(b_k^n)}{a_k^n - b_k^n} P_{n-1}b_k^n, z_0(a_k^n) \right\rangle.$$

By passing to the limit as $k \to \infty$ in the previous equality and by taking into account that $P_{n-1}b_k^n \to 0$ as $k \to \infty$ we obtain

$$\langle \dot{x}_0(\theta_0), z_0(\theta_0)\rangle = 0$$

which is a contradiction, see e.g. ([12], formula 12.9 Chap. III). Therefore, there exist $\Delta_0 > 0$ and $\delta_0 > 0$ such that $\Gamma : B_\Delta(C_\delta) \to \Gamma(B_\Delta(C_\delta))$ is injective for $\Delta \in (0, \Delta_0]$ and $\delta \in (0, \delta_0]$. Let us show that $\Delta_0 > 0$ and $\delta_0 > 0$ can be chosen also in such a way that

$$\Gamma(B_\Delta(C_\delta)) \ \text{ is open in } \mathbb{R}^n \text{ for any } \Delta \in (0, \Delta_0] \text{ and any } \delta \in (0, \delta_0]. \tag{2.4}$$

Observe that for any $\zeta \in \mathbb{R}^n$ satisfying $P_{n-1}\zeta = 0$ we have

$$\Gamma'(\zeta) = \frac{1}{\|Y\|_{M_T}} \Phi\left(\zeta^n + \overline{\theta}\right) \begin{pmatrix} \mathrm{e}^{\Lambda(\zeta^n + \overline{\theta})} & 0 \\ 0 & 0 \end{pmatrix} + \left(0 \ \ldots \ 0 \ \ \dot{x}_0(\zeta^n + \overline{\theta})\right)$$

and so for any $\zeta \in \mathbb{R}^n$ such that $P_{n-1}\zeta = 0$ the derivative $\Gamma'(\zeta)$ is invertible. Therefore, without loss of generality, we may consider $\Delta_0 > 0$ and $\delta_0 > 0$ sufficiently small to have that $\Gamma'(\zeta)$ is invertible for any $\zeta \in B_\Delta(C_\delta)$ with $\Delta \in (0, \Delta_0]$ and $\delta \in (0, \delta_0]$. By the inverse map theorem, see e.g. ([19], Theorem 9.17) we have that Γ is locally invertible in $B_\Delta(C_\delta)$ with $\Delta \in (0, \Delta_0]$ and $\delta \in (0, \delta_0]$, which implies that it maps any sufficiently small neighborhood of ζ in $\mathbb{R}^n$ into an open set of $\mathbb{R}^n$, which in turn implies (2.4). Moreover, from the inverse map theorem we have also that Γ^{-1} is continuously differentiable in $\Gamma(B_\Delta(C_\delta))$. □

We can now prove Theorem 2.1.

Proof. First of all observe that if x is a solution of the equation $x = Q_\varepsilon x$ then $u(t) = x^{-1}(t, x(t))$ is a solution of the equation $u = G_\varepsilon u$, see e.g. ([5], formulas (13)-(19)), where $G_\varepsilon : C([0,T], \mathbb{R}^n) \to C([0,T], \mathbb{R}^n)$ is defined as follows

$$(G_\varepsilon u)(t) = x(T, u(T)) + \varepsilon \int_0^t \left(x'_{(2)}(\tau, u(\tau))\right)^{-1} \phi(\tau, x(\tau, u(\tau)), \varepsilon) d\tau.$$

Moreover, since for any open set $V \subset \mathbb{R}^n$ the homeomorphism $(Mx)(t) = x^{-1}(t, x(t))$ maps every neighborhood of W_V onto a neighborhood of the set

$$\widehat{W}_V = \{u \in C([0,T], \mathbb{R}^n) : u(t) \in V, \text{ for any } t \in [0,T]\},$$

then by ([8], Theorem 26.4) we have that

$$d(I-Q_\varepsilon, W_{\Gamma(C_\delta)}) = d(I-G_\varepsilon, \widehat{W}_{\Gamma(C_\delta)})$$

provided that $d(I-G_\varepsilon, \widehat{W}_{\Gamma(C_\delta)})$ is defined. To show that $d(I-G_\varepsilon, \widehat{W}_{\Gamma(C_\delta)})$ is defined and to evaluate it, we introduce the vector field $A_\varepsilon : \Gamma(B_\Delta(C_\delta)) \to \mathbb{R}^n$ as follows

$$A_\varepsilon(\xi) = x'_{(2)}\left(T - \varepsilon f\left(\left[\Gamma^{-1}(\xi)\right]^n\right), x_0\left(\left[\Gamma^{-1}(\xi)\right]^n + \overline{\theta}\right)\right)\left(\xi - x_0\left(\left[\Gamma^{-1}(\xi)\right]^n + \overline{\theta}\right)\right) +$$

$$+ x_0\left(\left[\Gamma^{-1}(\xi)\right]^n + \overline{\theta} - \varepsilon f\left(\left[\Gamma^{-1}(\xi)\right]^n\right)\right),$$

where $\Gamma, \Delta, \delta > 0$ are given by Lemma 2.3 and $f : \mathbb{R} \to \mathbb{R}$ is defined as

$$f(t) = \begin{cases} |t|, & \text{if } f_{x_0}(\theta_1) < 0 \text{ and } f_{x_0}(\theta_2) < 0, \\ -|t|, & \text{if } f_{x_0}(\theta_1) > 0 \text{ and } f_{x_0}(\theta_2) > 0, \\ -d_{\mathbb{R}}\left(f_{x_0}, (\theta_1, \theta_2)\right) \cdot t, & \text{otherwise.} \end{cases} \tag{2.5}$$

We now prove that there exists $\delta_0 > 0$ such that for any $\delta \in (0, \delta_0]$ and any $\varepsilon \in (0, \delta^{1+\alpha}]$ both the topological degrees $d(I-G_\varepsilon, \widehat{W}_{\Gamma(C_\delta)})$ and $d_{\mathbb{R}^n}(I-A_\varepsilon, \Gamma(C_\delta))$ are defined and

$$d(I-G_\varepsilon, \widehat{W}_{\Gamma(C_\delta)}) = d_{\mathbb{R}^n}(I-A_\varepsilon, \Gamma(C_\delta)). \tag{2.6}$$

To do this we introduce an auxiliary vector field $\widehat{A}_\varepsilon : C([0,T], \mathbb{R}^n) \to C([0,T], \mathbb{R}^n)$ by letting $(\widehat{A}_\varepsilon u)(t) = A_\varepsilon(u(T))$ for any $t \in [0,T]$ and any $u \in C([0,T], \mathbb{R}^n)$. Since $\widehat{W}_{\Gamma(C_\delta)} \cap \mathbb{R}^n = \Gamma(C_\delta)$, by the reduction theorem for the topological degree, see e.g. ([8], Theorem 27.1), $d_{\mathbb{R}^n}(I-A_\varepsilon, \Gamma(C_\delta))$ is defined provided that $d(I-\widehat{A}_\varepsilon, \widehat{W}_{\Gamma(C_\delta)})$ is defined, moreover $d_{\mathbb{R}^n}(I-A_\varepsilon, \Gamma(C_\delta)) = d(I-\widehat{A}_\varepsilon, \widehat{W}_{\Gamma(C_\delta)})$. Hence, we now show that there exists $\delta_0 > 0$ such that for any $\delta \in (0, \delta_0]$ and any $\varepsilon \in (0, \delta^{1+\alpha}]$ both the Leray-Schauder topological degrees $d(I-G_\varepsilon, \widehat{W}_{\Gamma(C_\delta)})$ and $d(I-\widehat{A}_\varepsilon, \widehat{W}_{\Gamma(C_\delta)})$ are defined and

$$d(I-G_\varepsilon, \widehat{W}_{\Gamma(C_\delta)}) = d(I-\widehat{A}_\varepsilon, \widehat{W}_{\Gamma(C_\delta)}). \tag{2.7}$$

To prove (2.7) let $F_\varepsilon : C([0,T], \mathbb{R}^n) \to C([0,T], \mathbb{R}^n)$ be the operator given by

$$(F_\varepsilon u)(t) = \int_0^t \left(x'_{(2)}(\tau, u(\tau))\right)^{-1} \phi(\tau, x(\tau, u(\tau)), \varepsilon) d\tau \quad \text{for any } t \in [0,T],$$

and introduce the linear deformation

$$D_\varepsilon(\lambda, u)(t) = \lambda\Big(u(t) - x(T, u(T)) - \varepsilon(F_\varepsilon u)(t)\Big) + (1-\lambda)\left(u(t) - \left(\widehat{A}_\varepsilon u\right)(t)\right),$$

where $\lambda \in [0,1],\ u \in \partial\widehat{W}_{\Gamma(C_\delta)},\ \delta \in (0, \delta_0)$. Equivalently,

$$\begin{aligned} D_\varepsilon(\lambda, u)(t) = & \lambda\Big(u(t) - x(T, u(T))\Big) + (1-\lambda)u(t) \\ & - (1-\lambda)x'_{(2)}\left(T - \varepsilon f\left(\left[\Gamma^{-1}(u(T))\right]^n\right), \mathcal{P}_{x_0}(u(T))\right)\left(u(T) - \mathcal{P}_{x_0}(u(T))\right) \\ & - \lambda\varepsilon(F_\varepsilon u)(t) - (1-\lambda)\, x_0\left(\left[\Gamma^{-1}(u(T))\right]^n + \overline{\theta} - \varepsilon f\left(\left[\Gamma^{-1}(u(T))\right]^n\right)\right), \end{aligned}$$

where $\lambda \in [0,1],\ u \in \partial\widehat{W}_{\Gamma(C_\delta)},\ \delta \in (0, \delta_0)$ and

$$\mathcal{P}_{x_0}(\xi) = x_0\left(\left[\Gamma^{-1}(\xi)\right]^n + \overline{\theta}\right).$$

We show that for all sufficiently small $\delta \in (0,\delta_0]$ and $\varepsilon \in (0,\delta^{1+\alpha}]$ we have that $D_\varepsilon(\lambda,u) \neq 0$ for any $\lambda \in [0,1]$ and any $u \in \partial\widehat{W}_{\Gamma(C_\delta)}$. Assume the contrary, thus there exist $\{\delta_k\}_{k\in\mathbb{N}} \subset \mathbb{R}_+$, $\delta_k \to 0$ as $k \to \infty$, $\{\varepsilon_k\}_{k\in\mathbb{N}}$, $\varepsilon_k \in (0,\delta_k^{1+\alpha})$, $\{u_k\}_{k\in\mathbb{N}}$, $u_k \in \partial\widehat{W}_{\Gamma(C_{\delta_k})}$, $\{\lambda_k\}_{k\in\mathbb{N}} \subset [0,1]$ such that

$$\begin{aligned} 0 = {} & \lambda_k\Big(u_k(t) - x(T,u_k(T))\Big) + (1-\lambda_k)u_k(t) \\ & - (1-\lambda_k)x'_{(2)}\left(T - \varepsilon_k f\left(\left[\Gamma^{-1}(u_k(T))\right]^n\right), \mathcal{P}_{x_0}(u_k(T))\right)(u_k(T) - \mathcal{P}_{x_0}(u_k(T))) \\ & - \lambda_k\varepsilon_k(F_{\varepsilon_k}u_k)(t) - (1-\lambda_k)x_0\left(\left[\Gamma^{-1}(u_k(T))\right]^n + \overline{\theta} - \varepsilon_k f\left(\left[\Gamma^{-1}(u_k(T))\right]^n\right)\right). \end{aligned} \tag{2.8}$$

From (2.8) we have

$$\begin{aligned} u_k(t) = {} & \lambda_k x(T,u_k(T)) \\ & + (1-\lambda_k)x'_{(2)}\left(T - \varepsilon_k f\left(\left[\Gamma^{-1}(u_k(T))\right]^n\right), \mathcal{P}_{x_0}(u_k(T))\right)(u_k(T) - \mathcal{P}_{x_0}(u_k(T))) \\ & + \lambda_k\varepsilon_k(F_{\varepsilon_k}u_k)(t) + (1-\lambda_k)x_0\left(\left[\Gamma^{-1}(u_k(T))\right]^n + \overline{\theta} - \varepsilon_k f\left(\left[\Gamma^{-1}(u_k(T))\right]^n\right)\right) \end{aligned}$$

and therefore

$$\dot{u}_k(t) = \lambda_k\varepsilon_k\left(x'_{(2)}(t,u_k(t))\right)^{-1}\phi(t,x(t,u_k(t)),\varepsilon_k). \tag{2.9}$$

It follows from (2.9) that without loss of generality we may assume that there exists $\xi_0 \in \mathbb{R}^n$ such that

$$u_k(t) \to \xi_0 \text{ as } k \to \infty$$

uniformly with respect to $t \in [0,T]$. Since $u_k(0) \in \Gamma(C_{\delta_k}) \in B_{\delta_k}(x_0([\theta_1,\theta_2]))$ then $\xi_0 \in x_0([\theta_1,\theta_2])$. Now, to get a contradiction, take $t = T$ and rewrite (2.8) as follows

$$\begin{aligned} 0 = {} & \lambda_k\left(u_k(T) - x(T,u_k(T))\right) + (1-\lambda_k)u_k(T) \\ & - (1-\lambda_k)x'_{(2)}\left(T - \varepsilon_k f\left(\left[\Gamma^{-1}(u_k(T))\right]^n\right), \mathcal{P}_{x_0}(u_k(T))\right)(u_k(T) - \mathcal{P}_{x_0}(u_k(T))) \\ & - \lambda_k\varepsilon_k(F_{\varepsilon_k}u_k)(T) - (1-\lambda_k)x_0\left(\left[\Gamma^{-1}(u_k(T))\right]^n + \overline{\theta} - \varepsilon_k f\left(\left[\Gamma^{-1}(u_k(T))\right]^n\right)\right) \\ = {} & \lambda_k\left(u_k(T) - x(T,u_k(T))\right) + (1-\lambda_k)\left(I - x'_{(2)}\left(T - \varepsilon_k f\left(\left[\Gamma^{-1}(u_k(T))\right]^n\right), \mathcal{P}_{x_0}(u_k(T))\right)\right) \\ & \cdot (u_k(T) - \mathcal{P}_{x_0}(u_k(T))) - \lambda_k\varepsilon_k(F_{\varepsilon_k}u_k)(T) + (1-\lambda_k)\mathcal{P}_{x_0}(u_k(T)) \\ & - (1-\lambda_k)x_0\left(\left[\Gamma^{-1}(u_k(T))\right]^n + \overline{\theta} - \varepsilon_k f\left(\left[\Gamma^{-1}(u_k(T))\right]^n\right)\right) = \lambda_k\left(u_k(T) - x(T,u_k(T))\right) \\ & + (1-\lambda_k)\left(I - x'_{(2)}\left(T - \varepsilon_k f\left(\left[\Gamma^{-1}(u_k(T))\right]^n\right), \mathcal{P}_{x_0}(u_k(T))\right)\right)(u_k(T) - \mathcal{P}_{x_0}(u_k(T))) \\ & - \lambda_k\varepsilon_k(F_{\varepsilon_k}u_k)(T) + \varepsilon_k(1-\lambda_k)\dot{x}_0\left(\left[\Gamma^{-1}(u_k(T))\right]^n + \overline{\theta}\right) f\left(\left[\Gamma^{-1}(u_k(T))\right]^n\right) + o(\varepsilon_k). \end{aligned} \tag{2.10}$$

Now, observing that

$$\begin{aligned} x(T,\xi) - \xi &= x(T,\xi) - \mathcal{P}_{x_0}(\xi) + \mathcal{P}_{x_0}(\xi) - \xi \\ &= x(T,\mathcal{P}_{x_0}(\xi) + (\xi - \mathcal{P}_{x_0}(\xi))) - \mathcal{P}_{x_0}(\xi) + \mathcal{P}_{x_0}(\xi) - \xi \\ &= x'_{(2)}(T,\mathcal{P}_{x_0}(\xi))(\xi - \mathcal{P}_{x_0}(\xi)) - (\xi - \mathcal{P}_{x_0}(\xi)) + o(\xi - \mathcal{P}_{x_0}(\xi)) \\ &= \left(x'_{(2)}(T,\mathcal{P}_{x_0}(\xi)) - I\right)(\xi - \mathcal{P}_{x_0}(\xi)) + o(\xi - \mathcal{P}_{x_0}(\xi)), \end{aligned}$$

from (2.10) we obtain

$$\begin{aligned}&\lambda_k\left(I-x'_{(2)}(T,\mathcal{P}_{x_0}(u_k(T)))\right)(u_k(T)-\mathcal{P}_{x_0}(u_k(T)))-\lambda_k\, o(u_k(T)-\mathcal{P}_{x_0}(u_k(T)))\\&+(1-\lambda_k)\left(I-x'_{(2)}\left(T-\varepsilon_k f\left(\left[\Gamma^{-1}(u_k(T))\right]^n\right),\mathcal{P}_{x_0}(u_k(T))\right)\right)(u_k(T)-\mathcal{P}_{x_0}(u_k(T)))\\&-\lambda_k\varepsilon_k(F_{\varepsilon_k}u_k)(T)+\varepsilon_k(1-\lambda_k)\dot{x}_0\left(\left[\Gamma^{-1}(u_k(T))\right]^n+\overline{\theta}\right)f\left(\left[\Gamma^{-1}(u_k(T))\right]^n\right)+o(\varepsilon_k)=0.\end{aligned}\tag{2.11}$$

We may assume that the sequences $\{\lambda_k\}_{k\in\mathbb{N}}$ and $\left\{\dfrac{u_k(T)-\mathcal{P}_{x_0}(u_k(T))}{\|u_k(T)-\mathcal{P}_{x_0}(u_k(T))\|}\right\}_{k\in\mathbb{N}}$ converge, let $\lambda_0=\lim_{k\to\infty}\lambda_k$ and $l_0=\lim_{k\to\infty}\dfrac{u_k(T)-\mathcal{P}_{x_0}(u_k(T))}{\|u_k(T)-\mathcal{P}_{x_0}(u_k(T))\|}$. Since $u_k\in\partial\widehat{W}_{\Gamma(C_{\delta_k})}$ then there exists $t_k\in[0,T]$ such that $u_k(t_k)\in\partial\Gamma(C_{\delta_k})$. Let $\zeta_k=\Gamma^{-1}(u_k(t_k))$, without loss of generality we may assume that either

$$\zeta_k^n+\overline{\theta}\in(\theta_1,\theta_2)\quad\text{for any }k\in\mathbb{N}\tag{2.12}$$

or

$$\zeta_k^n+\overline{\theta}\in\{\theta_1\}\cup\{\theta_2\}\quad\text{for any }k\in\mathbb{N}.\tag{2.13}$$

Let us show that (2.12) cannot occur. By Lemma 2.3, Γ is a homeomorphism of $B_\Delta(C_{\delta_k})$ onto $\Gamma(B_\Delta(C_{\delta_k}))$ for sufficiently small $\Delta>0$ and $u_k(t_k)\in\partial\Gamma(C_{\delta_k})$ then we have

$$\zeta_k=\Gamma^{-1}(u_k(t_k))\in\partial C_{\delta_k}.\tag{2.14}$$

Hence (2.12) and (2.14) imply

$$\|P_{n-1}\zeta_k\|=\delta_k\quad\text{for any }k\in\mathbb{N}.\tag{2.15}$$

Since

$$\|P_{n-1}\zeta_k\|=\|Y^{-1}(\theta)Y(\theta)P_{n-1}\zeta_k\|\le\|Y^{-1}(\theta)\|\|Y(\theta)P_{n-1}\zeta_k\|$$

then there exists $c>0$ such that

$$\|Y(\theta)P_{n-1}\zeta_k\|\ge c\|P_{n-1}\zeta_k\|=c\delta_k$$

for any $\theta\in[0,T]$, and so we have

$$\|u_k(t_k)-\mathcal{P}_{x_0}(u_k(t_k))\|=\|\Gamma(\zeta_k)-x_0(\zeta_k^n+\overline{\theta})\|=\left\|Y\left(\zeta_k^n+\overline{\theta}\right)P_{n-1}\zeta_k\right\|\ge c\delta_k\tag{2.16}$$

for any $k\in\mathbb{N}$. On the other hand from (2.9) we have that there exists $c_1>0$ such that

$$\|u_k(T)-u_k(t_k)\|\le c_1\varepsilon_k\quad\text{for any }k\in\mathbb{N}.\tag{2.17}$$

Finally, from Lemma 2.3 we have that $x_0\left(\left[\Gamma^{-1}(\cdot)\right]^n+\overline{\theta}\right)$ is continuously differentiable and so by taking into account (2.17) there exists $c_2>0$ such that

$$\begin{aligned}&\|\mathcal{P}_{x_0}(u_k(T))-\mathcal{P}_{x_0}(u_k(t_k))\|\\&=\left\|x_0\left(\left[\Gamma^{-1}(u_k(T))\right]^n+\overline{\theta}\right)-x_0\left(\left[\Gamma^{-1}(u_k(t_k))\right]^n+\overline{\theta}\right)\right\|\\&\le c_2\|u_k(T)-u_k(t_k)\|\le c_1c_2\varepsilon_k\quad\text{for any }k\in\mathbb{N}.\end{aligned}\tag{2.18}$$

We are now in a position to estimate $\|u_k(T)-\mathcal{P}_{x_0}(u_k(T))\|$ from below. We have

$$\begin{aligned}&\|u_k(T)-\mathcal{P}_{x_0}(u_k(T))\|\\&=\|u_k(t_k)-\mathcal{P}_{x_0}(u_k(t_k))+u_k(T)-u_k(t_k)-(\mathcal{P}_{x_0}(u_k(T))-\mathcal{P}_{x_0}(u_k(t_k)))\|\\&\ge\left|\|u_k(t_k)-\mathcal{P}_{x_0}(u_k(t_k))\|-\|u_k(T)-u_k(t_k)-(\mathcal{P}_{x_0}(u_k(T))-\mathcal{P}_{x_0}(u_k(t_k)))\|\right|.\end{aligned}\tag{2.19}$$

Since $\varepsilon_k \in (0, \delta_k^{1+\alpha})$ there exists $k_0 \in \mathbb{N}$ such that $c_1\varepsilon_k + c_1c_2\varepsilon_k < c\delta_k$ for all $k \geq k_0$. Therefore, from (2.17) and (2.18) we have

$$\|u_k(T) - u_k(t_k) - (\mathcal{P}_{x_0}(u_k(T)) - \mathcal{P}_{x_0}(u_k(t_k)))\| \leq c_1\varepsilon_k + c_1c_2\varepsilon_k < c\delta_k, \tag{2.20}$$

for any $k \geq k_0$. By using (2.16) and (2.20) we may rewrite (2.19) as follows

$$\begin{aligned} &\|u_k(T) - \mathcal{P}_{x_0}(u_k(T))\| \\ &\geq \|u_k(t_k) - \mathcal{P}_{x_0}(u_k(t_k))\| \\ &- \|u_k(T) - u_k(t_k) - (\mathcal{P}_{x_0}(u_k(T)) - \mathcal{P}_{x_0}(u_k(t_k)))\| \end{aligned} \tag{2.21}$$

and so

$$\|u_k(T) - \mathcal{P}_{x_0}(u_k(T))\| \geq c\delta_k - c_1\varepsilon_k - c_1c_2\varepsilon_k \quad \text{for any } k \geq k_0.$$

By using this inequality we obtain for any $k \geq k_0$

$$\begin{aligned} &\frac{\varepsilon_k}{\|u_k(T) - \mathcal{P}_{x_0}(u_k(T))\|} \leq \frac{\varepsilon_k}{c\delta_k - c_1\varepsilon_k - c_1c_2\varepsilon_k} \leq \\ &\leq \frac{\delta_k^{1+\alpha}}{c\delta_k - c_1\delta_k^{1+\alpha} - c_1c_2\delta_k^{1+\alpha}} = \frac{\delta_k^{\alpha}}{c - c_1\delta_k^{\alpha} - c_1c_2\delta_k^{\alpha}}. \end{aligned} \tag{2.22}$$

Using (2.22) and passing to the limit as $k \to \infty$ in (2.11) divided by $\|u_k(T) - \mathcal{P}_{x_0}(u_k(T))\|$ we get

$$\left(I - x'_{(2)}(T, x_0(\zeta_0^n + \overline{\theta}))\right) l_0 = 0. \tag{2.23}$$

In order to prove that (2.23) leads to a contradiction we now show that

$$\left\langle (I - x'_{(2)}(T, \xi_0))l_0, z_0\left(\left[\Gamma^{-1}(\xi_0)\right]^n + \overline{\theta}\right)\right\rangle = 0. \tag{2.24}$$

Indeed

$$\begin{aligned} &\left\langle \frac{u_k(T) - \mathcal{P}_{x_0}(u_k(T))}{\|u_k(T) - \mathcal{P}_{x_0}(u_k(T))\|}, z_0\left(\left[\Gamma^{-1}(u_k(T))\right]^n + \overline{\theta}\right)\right\rangle \\ &= \frac{1}{\|u_k(T) - \mathcal{P}_{x_0}(u_k(T))\|}\left\langle \Gamma(\Gamma^{-1}(u_k(T))) - x_0\left(\left[\Gamma^{-1}(u_k(T))\right]^n + \overline{\theta}\right), z_0\left(\left[\Gamma^{-1}(u_k(T))\right]^n + \overline{\theta}\right)\right\rangle \\ &= \frac{1}{\|u_k(T) - \mathcal{P}_{x_0}(u_k(T))\|}\left\langle Y\left(\left[\Gamma^{-1}(u_k(T))\right]^n + \overline{\theta}\right) P_{n-1}\Gamma^{-1}(u_k(T)), z_0\left(\left[\Gamma^{-1}(u_k(T))\right]^n + \overline{\theta}\right)\right\rangle, \end{aligned}$$

and so by Lemma 2.2 we can conclude that

$$\left\langle \frac{u_k(T) - \mathcal{P}_{x_0}(u_k(T))}{\|u_k(T) - \mathcal{P}_{x_0}(u_k(T))\|}, z_0\left(\left[\Gamma^{-1}(u_k(T))\right]^n + \overline{\theta}\right)\right\rangle = 0 \quad \text{for any } k \in \mathbb{N}. \tag{2.25}$$

By the definition of the vector l_0 from (2.25), passing to the limit as $k \to \infty$, we obtain

$$\left\langle l_0, z_0\left(\zeta_0^n + \overline{\theta}\right)\right\rangle = 0. \tag{2.26}$$

Since $\|l_0\| = 1$ and so $l_0 \neq 0$, from Lemma 2.2 we have that there exists $l_* \neq 0$ such that

$$l_0 = Y\left(\zeta_0^n + \overline{\theta}\right) P_{n-1}l_* \quad \text{and} \quad P_{n-1}l_* = l_*, \tag{2.27}$$

observing that, see e.g. ([9], Theorem 2.1),

$$x'_{(2)}(t, x_0(\tau)) = Y(t+\tau)Y^{-1}(\tau), \quad \text{for any } t, \tau \in \mathbb{R} \tag{2.28}$$

we have

$$\begin{aligned}
&\left(I - x'_{(2)}(T, x_0(\zeta_0^n + \overline{\theta}))\right) l_0 = \left(I - Y\left(T + \zeta_0^n + \overline{\theta}\right) Y^{-1}\left(\zeta_0^n + \overline{\theta}\right)\right) l_0 \\
&= \left(Y\left(\zeta_0^n + \overline{\theta}\right) - Y\left(T + \zeta_0^n + \overline{\theta}\right)\right) P_{n-1} l_* \\
&= \Phi\left(\zeta_0^n + \overline{\theta}\right)\left(\begin{pmatrix} \mathrm{e}^{\Lambda(\zeta_0^n + \overline{\theta})} & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} \mathrm{e}^{\Lambda(T + \zeta_0^n + \overline{\theta})} & 0 \\ 0 & 1 \end{pmatrix}\right) P_{n-1} l_* \\
&= \Phi\left(\zeta_0^n + \overline{\theta}\right)\begin{pmatrix} \mathrm{e}^{\Lambda(\zeta_0^n + \overline{\theta})}(I - \mathrm{e}^{\Lambda T}) & 0 \\ 0 & 0 \end{pmatrix} P_{n-1} l_*
\end{aligned} \tag{2.29}$$

contradicting (2.23).

Let us now show that (2.13) also cannot occur. Firstly observe that if, passing to a subsequence if necessary, we have that $\dfrac{\varepsilon_k}{\|u_k(T) - \mathcal{P}_{x_0}(u_k(T))\|} \to 0$ then we can proceed as before to obtain again (2.23) and so a contradiction. Therefore, consider the case when $\dfrac{\varepsilon_k}{\|u_k(T) - \mathcal{P}_{x_0}(u_k(T))\|} \to l$, with $l > 0$ or $l = +\infty$. From (2.11) we have that

$$\begin{aligned}
&\frac{\varepsilon_k}{\|u_k(T) - \mathcal{P}_{x_0}(u_k(T))\|}\left\langle \Xi_k(x_0)(T), z_0\left(\left[\Gamma^{-1}(u_k(T))\right]^n + \overline{\theta}\right)\right\rangle \\
&= \left\langle \Upsilon_k(x_0)(T), z_0\left(\left[\Gamma^{-1}(u_k(T))\right]^n + \overline{\theta}\right)\right\rangle,
\end{aligned} \tag{2.30}$$

where

$$\Xi_k(x_0)(T) := \lambda_k(F_{\varepsilon_k} u_k)(T) - (1 - \lambda_k)\dot{x}_0\left(\left[\Gamma^{-1}(u_k(T))\right]^n + \overline{\theta}\right) f\left(\left[\Gamma^{-1}(u_k(T))\right]^n\right) + \frac{o(\varepsilon_k)}{\varepsilon_k},$$

$$\begin{aligned}
\Upsilon_k(x_0)(T) :=& \lambda_k\left(I - x'_{(2)}(T, \mathcal{P}_{x_0}(u_k(T)))\right)\frac{u_k(T) - \mathcal{P}_{x_0}(u_k(T)))}{\|u_k(T) - \mathcal{P}_{x_0}(u_k(T))\|} - \\
&- \lambda_k \frac{o(u_k(T) - \mathcal{P}_{x_0}(u_k(T)))}{\|u_k(T) - \mathcal{P}_{x_0}(u_k(T))\|} + (1 - \lambda_k)\frac{u_k(T) - \mathcal{P}_{x_0}(u_k(T))}{\|u_k(T) - \mathcal{P}_{x_0}(u_k(T))\|} \cdot \\
&\cdot \left(I - x'_{(2)}\left(T - \varepsilon_k f\left(\left[\Gamma^{-1}(u_k(T))\right]^n\right), \mathcal{P}_{x_0}(u_k(T))\right)\right).
\end{aligned}$$

By using (2.27), (2.29) and Lemma 2.2 we obtain

$$\begin{aligned}
&\left\langle\left(I - x'_{(2)}\left(T, x_0\left(\zeta_0^n + \overline{\theta}\right)\right)\right) l_0, z_0\left(\zeta_0^n + \overline{\theta}\right)\right\rangle = \left\langle Y\left(\zeta_0^n + \overline{\theta}\right)\left(I - \mathrm{e}^{\Lambda T}\right) P_{n-1} l_*, z_0\left(\zeta_0^n + \overline{\theta}\right)\right\rangle \\
&= \left\langle Y\left(\zeta_0^n + \overline{\theta}\right) P_{n-1}\left(I - \mathrm{e}^{\Lambda T}\right) l_*, z_0\left(\zeta_0^n + \overline{\theta}\right)\right\rangle = 0.
\end{aligned}$$

Therefore

$$\left\langle \Upsilon_k(x_0)(T), z_0\left(\left[\Gamma^{-1}(u_k(T))\right]^n + \overline{\theta}\right)\right\rangle \to 0 \quad \text{as } k \to \infty$$

and from (2.30) we conclude that

$$\left\langle \Xi_k(x_0)(T), z_0\left(\left[\Gamma^{-1}(u_k(T))\right]^n + \overline{\theta}\right)\right\rangle \to 0 \quad \text{as } k \to \infty$$

which imply

$$\left\langle \lambda_0 \widehat{F}\left(x_0\left(\zeta_0^n + \overline{\theta}\right)\right) - (1 - \lambda_0)\dot{x}_0\left(\zeta_0^n + \overline{\theta}\right) f\left(\zeta_0^n\right), z_0\left(\zeta_0^n + \overline{\theta}\right)\right\rangle = 0, \tag{2.31}$$

where

$$\widehat{F}(\xi) = \int_0^T \left(x'_{(2)}(\tau, \xi)\right)^{-1} \phi(\tau, x(\tau, \xi), 0) d\tau.$$

By Perron's lemma we have

$$\left\langle \dot{x}_0\left(\zeta_0^n+\overline{\theta}\right) f\left(\zeta_0^n\right), z_0\left(\zeta_0^n+\overline{\theta}\right)\right\rangle = \langle \dot{x}_0(0), z_0(0)\rangle f\left(\zeta_0^n\right)$$

and so (2.31) can be rewritten as

$$\lambda_0 \operatorname{sign}\langle \dot{x}_0(0), z_0(0)\rangle \left\langle \widehat{F}\left(x_0(\zeta_0^n+\overline{\theta})\right), z_0(\zeta_0^n+\overline{\theta})\right\rangle - (1-\lambda_0)\left|\langle \dot{x}_0(0), z_0(0)\rangle\right| f\left(\zeta_0^n\right) = 0, \quad (2.32)$$

let us show that

$$\operatorname{sign}\langle \dot{x}_0(0), z_0(0)\rangle \left\langle \widehat{F}\left(x_0\left(\theta\right)\right), z_0\left(\theta\right)\right\rangle = f_{x_0}(\theta) \quad \text{for any } \theta\in[0,T]. \quad (2.33)$$

Denote by $Z(t)$ and $Z_0(t)$ the fundamental matrixes of the adjoint system (1.12) such that $Z(0)=I$ and $Z_0(t)=(Z_{n-1}(t)\ z_0(t))$, where $Z_{n-1}(t)$ is a $n\times n-1$ matrix whose columns are (not T-periodic) linearly independent solutions of (1.12). Since

$$\left(x'_{(2)}(\tau, x_0(\theta))\right)^{-1} = Y(\theta)Y^{-1}(\tau+\theta) = \left(Z^{-1}(\theta)\right)^* Z^*(\tau+\theta) = \left(Z_0^{-1}(\theta)\right)^* Z_0^*(\tau+\theta),$$

see e.g. ([4], Chap. III §12), and $z_0(\theta) = (Z_{n-1}(\theta)\ \ z_0(\theta))\begin{pmatrix}0\\ \vdots\\ 0\\ 1\end{pmatrix}$ then we have

$$\begin{aligned}
\left\langle \widehat{F}\left(x_0\left(\theta\right)\right), z_0\left(\theta\right)\right\rangle &= \left\langle \left(Z_0^{-1}(\theta)\right)^* \int_0^T Z_0^*(\tau+\theta)\,\phi(\tau, x_0(\tau+\theta),0)d\tau, z_0\left(\theta\right)\right\rangle \\
&= \left\langle \int_\theta^{T+\theta} \begin{pmatrix} Z_{n-1}^*(\tau) \\ z_0(\tau)\end{pmatrix} \phi(\tau-\theta, x_0(\tau),0)d\tau, \begin{pmatrix}0\\ \vdots\\ 0\\ 1\end{pmatrix}\right\rangle \\
&= \int_\theta^{T+\theta} \langle z_0(\tau),\ \phi(\tau-\theta, x_0(\tau),0)\rangle\, d\tau = f_{x_0}(\theta)
\end{aligned}$$

and so (2.33) holds. By taking into account (2.33) we can finally rewrite (2.32) as follows

$$\lambda_0 f_{x_0}\left(\zeta_0^n+\overline{\theta}\right) - (1-\lambda_0)\left|\langle \dot{x}_0(0), z_0(0)\rangle\right| f\left(\zeta_0^n\right) = 0,$$

where either $\zeta_0^n+\overline{\theta}=\theta_1$ or $\zeta_0^n+\overline{\theta}=\theta_2$. This can be rewritten as

$$\lambda_0 f_{x_0}\left(\theta_i\right) - (1-\lambda_0)\left|\langle \dot{x}_0(0), z_0(0)\rangle\right| f\left((-1)^i|\zeta_0^n|\right) = 0, \quad (2.34)$$

where either $i=1$ or $i=2$. If $d_{\mathbb{R}}(f_{x_0},(\theta_1,\theta_2))=0$, then, see ([8], §3.2) for the definition of Brouwer degree in $\mathbb{R}$, for any $i=1,2$ and any $a\geq 0$ we have

$$f\left((-1)^i a\right) = -a\operatorname{sign}(f_{x_0}(\theta_1)) = -a\operatorname{sign}(f_{x_0}(\theta_2))$$

and so if $d_{\mathbb{R}}(f_{x_0},(\theta_1,\theta_2))=0$ then (2.34) can be rewritten as

$$\lambda_0 f_{x_0}\left(\theta_i\right) + (1-\lambda_0)\left|\langle \dot{x}_0(0), z_0(0)\rangle\right|\ |\zeta_0^n|\operatorname{sign}\left(f_{x_0}(\theta_i)\right) = 0, \quad (2.35)$$

where either $i=1$ or $i=2$. If $d_{\mathbb{R}}(f_{x_0},(\theta_1,\theta_2))\neq 0$, then for $i=1,2$ and any $a\geq 0$ we have

$$f\left((-1)^i a\right) = (-1)^{i+1}a\, d_{\mathbb{R}}(f_{x_0},(\theta_1,\theta_2)) = (-1)^{i+1}a\,(-1)^i\operatorname{sign}(f_{x_0}(\theta_i)) = -a\operatorname{sign}(f_{x_0}(\theta_i))$$

and so (2.34) can be rewritten again as (2.35). But (2.35) contradicts either the assumption that $f_{x_0}(\theta_1) \neq 0$ (in the case when $i = 1$) or the assumption that $f_{x_0}(\theta_2) \neq 0$ (in the case when $i = 2$).

Therefore, neither (2.13) nor (2.12) can occur and so there exists $\delta_0 > 0$ such that for any $\delta \in (0, \delta_0]$ and any $\varepsilon \in (0, \delta^{1+\alpha}]$ we have that $D_\varepsilon(\lambda, u) \neq 0$ for any $\lambda \in [0,1]$ and any $u \in \partial \widehat{W}_{\Gamma(C_\delta)}$. Thus for any $\delta \in (0, \delta_0]$ and $\varepsilon \in (0, \delta^{1+\alpha}]$ both the Leray-Schauder degrees $d(I - G_\varepsilon, \widehat{W}_{\Gamma(C_\delta)})$ and $d(I - \widehat{A}_\varepsilon, \widehat{W}_{\Gamma(C_\delta)})$ are defined and (2.7) holds. As already noticed (2.7) implies (2.6), hence to finish the proof it remains only to show that $d(I - A_\varepsilon, \Gamma(C_\delta)) = (-1)^{\beta(x_0)} d_{\mathbb{R}}\left(f_{x_0}, (\theta_1, \theta_2)\right)$ for any $\delta \in (0, \delta_0]$ and $\varepsilon \in (0, \delta^{1+\alpha}]$. Let $\delta \in (0, \delta_0]$ and $\varepsilon \in (0, \delta^{1+\alpha}]$, since Γ is a homeomorphism of $B_\Delta(C_\delta)$ onto $\Gamma(B_\Delta(C_\delta))$ by ([8], Theorem 26.4) we obtain

$$d_{\mathbb{R}^n}(I - A_\varepsilon, \Gamma(C_\delta)) = d_{\mathbb{R}^n}(I - \Gamma^{-1} A_\varepsilon \Gamma, C_\delta).$$

Let $\zeta \in C_\delta$. Taking into account (2.28) and (2.29) we have

$$\begin{aligned}
\zeta - (\Gamma^{-1} A_\varepsilon \Gamma)(\zeta) &= \zeta - (\Gamma^{-1} A_\varepsilon) \left(\frac{Y(\zeta^n + \overline{\theta})}{\|Y\|_{M_T}} P_{n-1} \zeta + x_0(\zeta^n + \overline{\theta}) \right) \\
&= \zeta - \Gamma^{-1} \left(x'_{(2)} \left(T - \varepsilon f(\zeta^n), x_0\left(\zeta^n + \overline{\theta}\right)\right) \frac{Y(\zeta^n + \overline{\theta})}{\|Y\|_{M_T}} P_{n-1} \zeta + x_0(\zeta^n + \overline{\theta} - \varepsilon f(\zeta^n)) \right) \\
&= \zeta - \Gamma^{-1} \left(\frac{Y\left(\zeta^n + \overline{\theta} - \varepsilon f(\zeta^n)\right)}{\|Y\|_{M_T}} P_{n-1} \begin{pmatrix} \mathrm{e}^{\Lambda T} & 0 \\ 0 & 0 \end{pmatrix} \zeta + x_0\left(\zeta^n + \overline{\theta} - \varepsilon f(\zeta^n)\right) \right) \\
&= \zeta - \begin{pmatrix} \mathrm{e}^{\Lambda T}\left(\zeta|_{\mathbb{R}^{n-1}}\right) \\ \zeta^n - \varepsilon f(\zeta^n) \end{pmatrix}
\end{aligned}$$

and so

$$d_{\mathbb{R}^n}(I - \Gamma^{-1} A_\varepsilon \Gamma, C_\delta) = d_{\mathbb{R}^n}\left(\left(I - \mathrm{e}^{\Lambda T}\right) \times \varepsilon f, C_\delta\right),$$

where $\left(I - \mathrm{e}^{\Lambda T}\right) \times \varepsilon f = \left(I - \mathrm{e}^{\Lambda T}, \varepsilon f\right)$. By the property of the Brouwer topological degree for the product of vector fields, see e.g. ([8], Theorem 7.4) we have

$$d_{\mathbb{R}^n}\left(\left(I - \mathrm{e}^{\Lambda T}\right) \times \varepsilon f, C_\delta\right) = d_{\mathbb{R}^n}\left(I - \mathrm{e}^{\Lambda T}, B_\delta(0)\right) \cdot d_{\mathbb{R}}\left(\varepsilon f, \left(-\frac{\theta_2 - \theta_1}{2}, \frac{\theta_2 - \theta_1}{2}\right)\right),$$

where $d_{\mathbb{R}^n}\left(I - \mathrm{e}^{\Lambda T}, B_\delta(0)\right) = (-1)^{\beta(x_0)}$ by ([8], Theorem 6.1), and by a direct computation we have that

$$d_{\mathbb{R}}\left(\varepsilon f, \left(-\frac{\theta_2 - \theta_1}{2}, \frac{\theta_2 - \theta_1}{2}\right)\right) = -d_{\mathbb{R}}\left(f_{x_0}, (\theta_1, \theta_2)\right).$$

Thus, we finally have that

$$d_{\mathbb{R}^n}(I - \Gamma^{-1} A_\varepsilon \Gamma, C_\delta) = -(-1)^{\beta(x_0)} d_{\mathbb{R}}\left(f_{x_0}, (\theta_1, \theta_2)\right).$$

In conclusion, we have proved that there exists $\delta_0 > 0$ such that for any $\delta \in (0, \delta_0]$ and any $\varepsilon \in (0, \delta^{1+\alpha}]$ the Leray-Schauder topological degree $d(I - Q_\varepsilon, W_{\Gamma(C_\delta)})$ is defined and it can be calculated by the formula

$$d(I - Q_\varepsilon, W_{\Gamma(C_\delta)}) = -(-1)^{\beta(x_0)} d_{\mathbb{R}}\left(f_{x_0}, (\theta_1, \theta_2)\right).$$

To conclude the proof we have only to show that $V_\delta := \Gamma(C_\delta)$ satisfies properties 1) and 2). To this end, let $\xi \in \Gamma(C_\delta)$, thus

$$\xi = \frac{Y(\zeta^n + \overline{\theta})}{\|Y\|_{M_T}} P_{n-1} \zeta + x_0(\zeta^n + \overline{\theta}).$$

for some $\zeta \in \mathbb{R}^n$ satisfying $\|P_{n-1}\zeta\| \le \delta$ and $\left[\Gamma^{-1}(\xi)\right]^n + \overline{\theta} \in [\theta_1, \theta_2]$. Therefore

$$\left\|\xi - x_0\left(\left[\Gamma^{-1}(\xi)\right]^n + \overline{\theta}\right)\right\| = \left\|\frac{Y(\zeta^n + \overline{\theta})}{\|Y\|_{M_T}} P_{n-1}\zeta\right\| \le \|P_{n-1}\zeta\| \le \delta$$

and so property 1) holds. By the definition of the set C_δ we have that for any $\delta \in (0, \delta_0)$ both the points $\left(0, ..., 0, -\dfrac{\theta_2 - \theta_1}{2}\right)$ and $\left(0, ..., 0, \dfrac{\theta_2 - \theta_1}{2}\right)$ belong to the boundary of C_δ. Therefore, both the points $x_0(\theta_1)$ and $x_0(\theta_2)$ belong to the boundary of $\Gamma(C_\delta)$. On the other hand if $\xi = x_0(\theta)$, where $\theta \in (\theta_1, \theta_2)$, then

$$\Gamma^{-1}(\xi) = \left(0, ..., 0, \theta - \overline{\theta}\right) \subset C_\delta. \tag{2.36}$$

Thus $\xi \in \Gamma(C_\delta)$ and property 2) is also satisfied. The proof of Theorem 2.1 is now complete. □

Recall that

$$\Theta_W(x) = \{\theta_0 \in (0, T) : S_{\theta_0}\, x \in \partial W,\ S_\theta\, x \in W \text{ for any } \theta \in (0, \theta_0)\}, \quad \text{where } x \in \mathfrak{S}_W,$$

$$(S_\theta\, x)(t) = x(t + \theta) \quad \text{and}$$

$\beta(x_0)$ is the sum of the multiplicities of the characteristic multipliers greater than 1 of (1.10).

We can prove the following result.

Theorem 2.4 *Assume that $\mathfrak{S}_W$ is finite and it contains only nondegenerate T-periodic cycles of (1.8). Assume that $f_x(0) \neq 0$ for any $x \in \mathfrak{S}_W$. Then for every $\varepsilon > 0$ sufficiently small the topological degree $d(I - Q_\varepsilon, W)$ is defined and the following formula holds*

$$d(I - Q_\varepsilon, W) = (-1)^n d_{\mathbb{R}^n}(\psi, W \cap \mathbb{R}^n) - \sum_{x \in \mathfrak{S}_W :\ \Theta_W(x) \neq \emptyset} (-1)^{\beta(x)} d_{\mathbb{R}}\left(f_x, (0, \min\{\Theta_W(x)\})\right), \tag{2.37}$$

P r o o f. For any $x \in \mathfrak{S}_W$ satisfying $\Theta_W(x) \neq \emptyset$ let $\delta_0(x)$ and $\{V_\delta(x)\}_{\delta \in (0, \delta_0(x))}$ as given by Theorem 2.1, where $x_0 := x$, $\theta_1 := 0$ and $\theta_2 := \min\{\Theta_W(x)\}$. Let $\delta_1 = \min_{x \in \mathfrak{S}_W : \Theta_W(x) \neq \emptyset} \delta_0(x) > 0$. Since $f_x(0) \neq 0$ for any $x \in \mathfrak{S}_W$ then by Malkin's theorem, see [13] or ([12], Theorem p. 387), there exists $\delta_* \in (0, \delta_1)$ and $\varepsilon_* > 0$ such that

$$Q_\varepsilon \widetilde{x} \neq \widetilde{x} \quad \text{for any } \widetilde{x} \in \overline{B_{\delta_*}(x)} \text{ whenever } x \in \mathfrak{S}_W \text{ and } \varepsilon \in (0, \varepsilon_*). \tag{2.38}$$

By the definition of $\mathfrak{S}_W$ from (2.38) we have that

$$Q_\varepsilon \widetilde{x} \neq \widetilde{x} \quad \text{for any } \widetilde{x} \in \overline{B_{\delta_*}(x)} \cup \overline{B_{\delta_*}\left(S_{\min\{\Theta_W(x)\}} x\right)} \text{ whenever } x \in \mathfrak{S}_W \text{ and } \varepsilon \in (0, \varepsilon_*) \tag{2.39}$$

Let $\delta_{**} \in (0, \delta_*)$ be sufficiently small in such a way that

$$\left(B_{\delta_*}(x) \cup B_{\delta_*}(S_{\min\{\Theta_W(x)\}} x) \cup W_{V_{\delta_{**}}(x)}\right) \backslash \overline{W} \subset B_{\delta_*}(x) \cup B_{\delta_*}(S_{\min\{\Theta_W(x)\}} x)$$

for any $x \in \mathfrak{S}_W$, therefore by taking into account (2.39) we have

$$Q_\varepsilon \widetilde{x} \neq \widetilde{x} \quad \text{for any } \widetilde{x} \in \left(B_{\delta_*}(x) \cup B_{\delta_*}(S_{\min\{\Theta_W(x)\}} x) \cup W_{V_{\delta_{**}}(x)}\right) \backslash \overline{W},$$

whenever $x \in \mathfrak{S}_W$ and $\varepsilon \in (0, \varepsilon_*)$. Therefore by applying the coincidence degree formula given by Theorem 2.1 for any $x \in \mathfrak{S}_W$ such that $\Theta_W(x) \neq \emptyset$ and any $\varepsilon \in \left(0, \min\{\delta^{1+\alpha}, \varepsilon_*\}\right)$ we have

$$\begin{aligned} &d\left(I - Q_\varepsilon, \left(B_{\delta_*}(x) \cup B_{\delta_*}(S_{\min\{\Theta_W(x)\}} x) \cup W_{V_{\delta_{**}}(x)}\right) \cap W\right) \\ &= d\left(I - Q_\varepsilon, B_{\delta_*}(x) \cup B_{\delta_*}(S_{\min\{\Theta_W(x)\}} x) \cup W_{V_{\delta_{**}}(x)}\right) \\ &= d(I - Q_\varepsilon, W_{V_{\delta_{**}}(x)}) = -(-1)^{\beta(x)} d_{\mathbb{R}}\left(f_x, (0, \min\{\Theta_W(x)\})\right). \end{aligned} \tag{2.40}$$

Let

$$\mathfrak{S}_W^0 = \{x \in \mathfrak{S}_W : \text{there exists } \delta_0 > 0 \text{ such that } S_\delta(x) \notin \partial W \text{ for any } \delta \in (-\delta_0, 0) \cup (0, \delta_0)\}.$$

From (2.38) we have that

$$d\left(I - Q_\varepsilon, B_{\delta_*}(x) \cap W\right) = 0 \quad \text{for any } x \in \mathfrak{S}_W^0 \text{ and any } \varepsilon \in (0, \varepsilon_*). \tag{2.41}$$

Since any point $x \in \mathfrak{S}_W$ is a limit cycle of (1.8) and, by assumption, they are in a finite number we may assume without loss of generality that $\delta_* > 0$ is sufficiently small to have that

$$Q_0(\widehat{x}) \neq \widehat{x} \quad \text{for any } \widehat{x} \in C([0,T], \mathbb{R}^n) \text{ such that } \widehat{x}(0) \in B_{\delta_*}(x([0,T])) \backslash x([0,T]). \tag{2.42}$$

Therefore we have that the boundary of the set $W \backslash E_{\delta_*}$ where

$$E_{\delta_*} := \left(\bigcup_{x \in \mathfrak{S}_W : \Theta_W(x) \neq \emptyset} \left(B_{\delta_*}(x) \cup B_{\delta_*}(S_{\Theta_W(x)} x) \cup W_{V_{\delta_{**}}(x)}\right) \cap W \right) \bigcup \left(\bigcup_{x \in \mathfrak{S}_W^0} B_{\delta_*}(x) \cap W \right)$$

does not contain T-periodic solutions of (1.8). This fact allows us to apply Corollary 1 of [3] to obtain

$$d(I - Q_0, W \backslash E_{\delta_*}) = (-1)^n d_{\mathbb{R}^n}(\psi, E_{\delta_*} \cap \mathbb{R}^n). \tag{2.43}$$

But from (2.42) the function ψ is nondegenerate on the set $E_{\delta_*} \cap \mathbb{R}^n$ and from (2.43) we have that

$$d(I - Q_0, W \backslash E_{\delta_*}) = (-1)^n d_{\mathbb{R}^n}(\psi, W \cap \mathbb{R}^n). \tag{2.44}$$

From (2.40), (2.41) and (2.44) the conclusion of the theorem easily follows. □

Remark 2.5 From 2.37) it follows that the points of $\mathfrak{S}_W$ such that $S_\theta x \notin \mathfrak{S}_W$ for all $\theta \in (0,T)$ do not affect the value of $d(I - Q_\varepsilon, W)$ with $\varepsilon > 0$ sufficiently small.

Let $X = \{x \in C([0,T], \mathbb{R}^n) : x(0) = x(T)\}$ and let $L : \mathrm{dom} L \subset X \to L^1([0,T], \mathbb{R}^n)$ be the linear operator defined by $(Lx)(\cdot) = \dot{x}(\cdot)$ with $\mathrm{dom} L = \{x \in X : x(\cdot) \text{ is absolutely continuous}\}$. It is immediate to see that L is a Fredholm operator of index zero. Let $N_\varepsilon : X \to L^1([0,T], \mathbb{R}^n)$ be the Nemitcky operator given by $(N_\varepsilon x)(\cdot) = \psi(x(\cdot)) + \varepsilon \phi(\cdot, x(\cdot), \varepsilon)$. Thus the existence of T-periodic solutions for system (1.1) is equivalent to the solvability of the equation

$$Lx = N_\varepsilon x, \quad x \in \mathrm{dom} L. \tag{2.45}$$

We now provide for the coincidence degree $D_L(L - N_\varepsilon, W \cap X)$ of L and N_ε, see ([16], p. 19), a formula similar to that established in Theorem 2.1.

Corollary 2.6 *Assume all the conditions of Theorem 2.4. Then for $\varepsilon > 0$ sufficiently small the coincidence degree $D_L(L - N_\varepsilon, W \cap X)$ is defined and the following formula holds*

$$\begin{aligned} D_L(L - N_\varepsilon, W \cap X) &= (-1)^n d_{\mathbb{R}^n}(\psi, W \cap \mathbb{R}^n) \\ &\quad - \sum_{x \in \mathfrak{S}_W :\ \Theta_W(x) \neq \emptyset} (-1)^{\beta(x)} d_{\mathbb{R}}\left(f_x, (0, \min\{\Theta_W(x)\})\right). \end{aligned} \tag{2.46}$$

P r o o f. Since $d(I - Q_\varepsilon, W)$ is defined for $\varepsilon > 0$ sufficiently small then $D_L(L - N_\varepsilon, W \cap X)$ is also defined for $\varepsilon > 0$ sufficiently small, see ([16], Chap. 2 §2). To prove (2.46) we apply the duality principles developed in ([16], Chap. 3). First, observe that the zeros of the operator $R_\varepsilon : C([0,T], \mathbb{R}^n) \to C([0,T], \mathbb{R}^n)$ defined by

$$(R_\varepsilon x)(t) = x(t) - x(0) - \int_0^T \left(\psi(x(\tau)) + \varepsilon \phi(\tau, x(\tau), \varepsilon)\right) d\tau -$$

$$- \int_0^t \left(\psi(x(\tau)) + \varepsilon \phi(\tau, x(\tau), \varepsilon)\right) d\tau + t \int_0^T \left(\psi(x(\tau)) + \varepsilon \phi(\tau, x(\tau), \varepsilon)\right) d\tau$$

coincide with the fixed points of the operator Q_ε, hence $d(R_\varepsilon, W)$ is also defined for $\varepsilon > 0$ sufficiently small. Therefore by ([16], Theorem III.1 with $a = 1$ and $b = 0$) and ([16], Theorem III.4) we have that

$$d(R_\varepsilon, W) = d(I - Q_\varepsilon, W).$$

Furthermore, by using the methods employed in ([16], Chap. III, §4) for defining $D_L(L - N_\varepsilon, W \cap X)$ and by ([16], Theorem III.7) we obtain that

$$D_L(L - N_\varepsilon, W \cap X) = d(R_\varepsilon, W),$$

which concludes the proof. □

Remark 2.7 If $W = W_U$ for a suitable open set $U \subset \mathbb{R}^n$ then it is possible to rewrite (2.37) and (2.46) in a different way by representing the sets $\mathfrak{S}_W$ as follows

$$\mathfrak{S}_W = \bigcup_{\xi \in \partial U : x(0,\xi) = x(T,\xi)} x(\cdot, \xi)$$

and

$$\Theta_W(x) = \{\theta_0 \in (0,T) : x(\theta_0) \in \partial U,\ x(\theta) \in U \text{ for any } \theta \in (0,\theta_0)\}\,.$$

Moreover, if

$$\text{any Cauchy problem associated to (1.1) has an unique solution defined in } [0,T], \tag{2.47}$$

then we can introduce the Poincaré-Andronov operator $\Omega_\varepsilon : \mathbb{R}^n \to \mathbb{R}^n$ in the following way

$$\Omega_\varepsilon(\xi) = x_\varepsilon(T, \xi),$$

where $x_\varepsilon(\cdot, \xi)$ is the solution of (1.1) satisfying $x_\varepsilon(0,\xi) = \xi$. In this case we can provide an analogous result to (2.37) for the Brouwer topological degree of $I - \Omega_\varepsilon$ on U.

Indeed, we can prove the following result.

Corollary 2.8 *Assume that condition (2.47) is satisfied. Let*

$$\mathfrak{S}^U = \bigcup_{\xi \in \partial U : x(0,\xi) = x(T,\xi)} x(\cdot, \xi).$$

Assume that $\mathfrak{S}^U$ is finite and any T-periodic solution $x_0 \in \mathfrak{S}^U$ is a nondegenerate limit cycle of (1.8). If

$$f_x(0) \neq 0 \quad \text{for any } x \in \mathfrak{S}^U$$

then for all $\varepsilon > 0$ sufficiently small the topological degree $d_{\mathbb{R}^n}(I - \Omega_\varepsilon, U)$ is defined and it can be evaluated by the formula

$$d_{\mathbb{R}^n}(I - \Omega_\varepsilon, U) = (-1)^n d_{\mathbb{R}^n}(\psi, U) - \sum_{x \in \mathfrak{S}^U:\ \Theta^U(x) \neq \emptyset} (-1)^{\beta(x)} d_{\mathbb{R}} \left(f_x, \left(0, \min\{\Theta^U(x)\}\right)\right), \tag{2.48}$$

where, for any $x \in \mathfrak{S}^U$, $\Theta^U(x) = \{\theta_0 \in (0,T) : x(\theta_0) \in \partial U,\ x(\theta) \in U \textit{ for any } \theta \in (0,\theta_0)\}$ and $\beta(x)$ is the sum of the multiplicities of the characteristic multipliers greater than 1 of (1.10) with $x_0 := x$.

Proof. From Theorem 2.4, taking into account Remark 2.7 we have that there exists $\varepsilon_0 > 0$ such that for every $\varepsilon \in (0, \varepsilon_0]$ the degree $d(I - Q_\varepsilon, W_U)$ is defined and

$$\begin{aligned} d(I - Q_\varepsilon, W_U) =& (-1)^n d_{\mathbb{R}^n}(\psi, W_U \cap \mathbb{R}^n) \\ & - \sum_{x \in \mathfrak{S}^U:\ \Theta^U(x) \neq \emptyset} (-1)^{\beta(x)} d_{\mathbb{R}} \left(f_x, \left(0, \min\{\Theta^U(x)\}\right)\right). \end{aligned} \tag{2.49}$$

Therefore, to prove the corollary we show that

$$d(I - Q_\varepsilon, W_U) = d_{\mathbb{R}^n}(I - \Omega_\varepsilon, U) \quad \text{for any } \varepsilon \in (0, \varepsilon_0] \tag{2.50}$$

and

$$d_{\mathbb{R}^n}(\psi, W_U \cap \mathbb{R}^n) = d_{\mathbb{R}^n}(\psi, U). \tag{2.51}$$

To prove (2.50) let us define $W_U^\varepsilon \subset C([0,T], \mathbb{R}^n)$ as

$$W_U^\varepsilon = \left\{ \widehat{x} \in C([0,T], \mathbb{R}^n) : x_\varepsilon^{-1}(t, \widehat{x}(t)) \in U, \text{ for any } t \in [0,T] \right\}.$$

We claim that there exists $\widehat{\varepsilon}_0 \in (0, \varepsilon_0]$ such that

$$Q_\varepsilon x \neq x \quad \text{for any } x \in (W_U \backslash W_U^\varepsilon) \cup (W_U^\varepsilon \backslash W_U) \text{ and any } \varepsilon \in (0, \widehat{\varepsilon}_0]. \tag{2.52}$$

Assume the contrary, thus there exist sequences $\{\varepsilon_k\}_{k\in\mathbb{N}} \subset (0, \varepsilon_0]$, $\varepsilon_k \to 0$ as $k \to \infty$, $\{x_k\}_{k\in\mathbb{N}} \subset C([0,T], \mathbb{R}^n)$, such that

$$x_k \in (W_U \backslash W_U^{\varepsilon_n}) \cup (W_U^{\varepsilon_n} \backslash W_U), \tag{2.53}$$

and

$$x_k \to x_0 \text{ as } k \to \infty \text{ where } Q_{\varepsilon_k} x_k = x_k. \tag{2.54}$$

It is easy to see that (2.53) implies $x_0 \in \partial W_U$. This fact together with (2.54) and the assumption that $f_{x_0}(0) \neq 0$ leads to a contradiction with the Malkin's result (1.13)-(1.14). Therefore, we have proved that (2.52) holds and thus

$$d(I - Q_\varepsilon, W_U) = d(I - Q_\varepsilon, W_U^\varepsilon) \quad \text{for any } \varepsilon \in (0, \widehat{\varepsilon}_0].$$

Since for any $\varepsilon \geq 0$ the sets U and W_U^ε have a common core with respect to the T-periodic problem for system (1.1), see ([8], §28.5), then, by ([8], Theorem 28.5), we have

$$d(I - Q_\varepsilon, W_U^\varepsilon) = d_{\mathbb{R}^n}(I - \Omega_\varepsilon, U) \quad \text{for any } \varepsilon \geq 0$$

and so (2.50) is proved. Finally, the proof of (2.51) is obtained by means of the Leray-Schauder continuation principle. In fact, let

$$U_\lambda = \left\{ \xi \in \mathbb{R}^n : x^{-1}(\lambda t, \xi) \in U \text{ for any } t \in [0,T] \right\}, \quad \lambda \in [0,1],$$

we now show that

$$0 \notin \psi(\partial U_\lambda) \quad \text{for any } \lambda \in [0,1]. \tag{2.55}$$

Assume the contrary, thus there exists $\lambda_0 \in [0,1]$ such that $\xi_0 \in \partial U_{\lambda_0}$ and $\psi(\xi_0) = 0$. Observe, that $x^{-1}(\lambda_0 t, \xi_0) \in \overline{U}$ for any $t \in [0,T]$. Therefore, we have that there exists $t_0 \in [0,T]$ such that $x^{-1}(\lambda_0 t_0, \xi_0) \in \partial U$ and from the fact that $\psi(\xi_0) = 0$ we have that $x^{-1}(\lambda_0 t, \xi_0)$ is constant with respect to $t \in [0,T]$. Hence we have $x^{-1}(\lambda_0 t_0, \xi_0) = x^{-1}(0, \xi_0) = \xi_0$ and we obtain that $\xi_0 \in \partial U$ contradicting the fact that ∂U contains only initial conditions of nondegenerate limit cycles of (1.8). By using the Leray-Schauder continuation principle [10] (see also [2], Theorem 10.7) from (2.55) we now conclude that

$$d_{\mathbb{R}^n}(\psi, U_0) = d_{\mathbb{R}^n}(\psi, U_1).$$

On the other hand $U_0 = U$ and $U_1 = W_U \cap \mathbb{R}^n$ and so the proof of (2.51) is also complete. □

Remark 2.9 From (2.48) it follows that if the limit cycle $x \in \mathfrak{S}^U$ touches ∂U but it does not intersect ∂U then this cycle does not have any influence in the evaluation of $d_{\mathbb{R}^n}(I - \Omega_\varepsilon, W_U)$ with $\varepsilon > 0$ sufficiently small.

3 Existence of T-periodic solutions

By means of different choices of the set $W \subset C([0,T],\mathbb{R}^n)$ we formulate in what follows some existence results for T-periodic solutions to (1.1) in W.

Theorem 3.1 *Assume that all the nonconstant T-periodic solutions of (1.8) are nondegenerate limit cycles of (1.8). Then for any open bounded set $W \subset C([0,T],\mathbb{R}^n)$ containing all the constant solutions of (1.8) and satisfying the conditions*

$$\mathfrak{S}_W \textit{ is finite}, \quad f_x(0) \neq 0 \quad \textit{for any } x \in \mathfrak{S}_W$$

and

$$(-1)^n d_{\mathbb{R}^n}(\psi, W \cap \mathbb{R}^n) - \sum_{x \in \mathfrak{S}_W:\ \Theta_W(x) \neq \emptyset} (-1)^{\beta(x)} d_{\mathbb{R}}\left(f_x, (0, \min\{\Theta_W(x)\})\right) \neq 0$$

there exists $\varepsilon_0 > 0$ such that for any $\varepsilon \in (0, \varepsilon_0]$ system (1.1) has a T-periodic solution belonging to W.

The assumptions of Theorem 3.1 implies that the set $\mathfrak{S}_W$ contains only nondegenerate cycles of (1.8). Therefore, Theorem 3.1 follows from Theorem 2.4 and the solution property of the Leray-Schauder topological degree, see ([8], Theorem 20.5). Observe that Theorem 3.1 is an extension of ([3], Corollary 4).

The next result provides conditions under which the conclusion of ([3], Theorem 2) remains valid also in the case when ∂W contains T-periodic solutions to (1.8).

Corollary 3.2 *Assume that all the nonconstant T-periodic solutions of (1.8) are nondegenerate limit cycles of (1.8). Assume that there exists an open bounded set $W \subset C([0,T],\mathbb{R}^n)$ containing all the constant solutions of (1.8) and satisfying the conditions*

$$\mathfrak{S}_W \textit{ is finite}, \quad f_x(0) \cdot f_x(\min\{\Theta_W(x)\}) > 0 \quad \textit{for any } x \in \mathfrak{S}_W \textit{ with } \Theta_W(x) \neq \emptyset \tag{3.1}$$

and

$$d_{\mathbb{R}^n}(\psi, W \cap \mathbb{R}^n) \neq 0.$$

Then for any $\varepsilon > 0$ sufficiently small system (1.1) has a T-periodic solution belonging to W.

The proof of the Corollary 3.2 follows directly from the fact that (3.1) implies that

$$d_{\mathbb{R}}\left(f_x, (0, \min\{\Theta_W(x)\})\right) = 0 \quad \text{for any } x \in \mathfrak{S}_W \text{ with } \Theta_W(x) \neq \emptyset$$

(see [8], §3.2).

In what follows we give some applications of Theorem 2.1 to the problem of the existence of T-periodic solutions to (1.1) near a nondegenerate limit cycle of (1.8). In the sequel $\rho(\xi, A)$ denotes the distance between $\xi \in \mathbb{R}^n$ and $A \subset \mathbb{R}^n$ given by $\rho(\xi, A) = \inf_{\zeta \in A} \|\xi - \zeta\|$. First, we state the following result.

Theorem 3.3 *Let x_0 be a nondegenerate T-periodic limit cycle of (1.8). Let $0 \leq \theta_1 < \theta_2 \leq \theta_1 + \frac{T}{p}$ where $p \in \mathbb{N}$ and $\frac{T}{p}$ is the least period of x_0. Assume that*

$$f_{x_0}(\theta_1) \cdot f_{x_0}(\theta_2) < 0. \tag{3.2}$$

Let Θ be the set of all zeros of f_{x_0} on (θ_1, θ_2). Then, for any $\varepsilon > 0$ sufficiently small, system (1.1) has a T-periodic solution x_ε such that for any $t \in [0,T]$ we have

$$\rho\left(x_\varepsilon(t), x_0(t + \Theta)\right) \to 0 \quad \text{as } \varepsilon \to 0. \tag{3.3}$$

P r o o f. Observe, that condition (3.2) implies that

$$f_{x_0}(\theta_1) \neq 0 \quad \text{and} \quad f_{x_0}(\theta_2) \neq 0$$

and so the assumptions of Theorem 2.1 are satisfied. Let us fix $\alpha > 0$, from Theorem 2.1 we have that there exists $\delta_0 > 0$ such that for any $\varepsilon \in (0, \delta_0^{1+\alpha})$ the topological degree $d(I - Q_\varepsilon, W_{V_{\delta(\varepsilon)}})$ is defined with $\delta(\varepsilon) = \varepsilon^{1/(1+\alpha)}$ and

$$d(I - Q_\varepsilon, W_{V_{\delta(\varepsilon)}}) = -(-1)^{\beta(x_0)} d_{\mathbb{R}}(f_{x_0}, (\theta_1, \theta_2)).$$

From (3.2) we also have, see ([8], §3.2), that $|d_{\mathbb{R}}(f_{x_0}, (\theta_1, \theta_2))| = 1$ and so for any $\varepsilon \in (0, \delta_0^{1+\alpha})$ system (1.1) has a T-periodic solution x_ε such that $x_\varepsilon(0) \in V_{\delta(\varepsilon)}$. Moreover, from property 1) of Theorem 2.1 we have that

$$\rho\left(x_\varepsilon(0), x_0([\theta_1, \theta_2])\right) \le \delta(\varepsilon) = \varepsilon^{1/(1+\alpha)}. \tag{3.4}$$

Let $u_\varepsilon(t) = x^{-1}(t, x_\varepsilon(t))$, then, see e.g. ([5], (13)-(19)),

$$\dot{u}_\varepsilon(t) = \varepsilon \left(x'_{(2)}(t, u_\varepsilon(t))\right)^{-1} \phi(t, x(t, u_\varepsilon(t), \varepsilon)).$$

Therefore there exists $M_1 > 0$ such that

$$\|u_\varepsilon(0) - u_\varepsilon(t)\| \le M_1 \varepsilon \quad \text{for any } \varepsilon \in (0, \delta_0^{1+\alpha}) \text{ and any } t \in [0, T]. \tag{3.5}$$

On the other hand, $u_\varepsilon(0) = x_\varepsilon(0)$ and so from (3.4) and (3.5) for any $\varepsilon \in (0, \delta_0^{1+\alpha})$ and any $t \in [0, T]$ we have that

$$\rho(u_\varepsilon(t), x_0([\theta_1, \theta_2])) \le \|u_\varepsilon(t) - x_\varepsilon(0)\| + \rho(x_\varepsilon(0), x_0([\theta_1, \theta_2])) \le \varepsilon^{1/(1+\alpha)} \left(1 + M_1 \varepsilon^{\alpha/(1+\alpha)}\right). \tag{3.6}$$

Since for any $\theta \in [\theta_1, \theta_2]$ we have that $\|x_\varepsilon(t) - x_0(t + \theta)\| = \|x(t, u_\varepsilon(t)) - x(t, x_0(\theta))\|$ and since, as it was already observed, in the proof of Theorem 2.1, the function $x(\cdot, \cdot)$ is continuously differentiable with respect to both variables we have that there exists $M_2 > 0$ such that

$$\|x_\varepsilon(t) - x_0(t + \theta)\| \le M_2 \|u_\varepsilon(t) - x_0(\theta)\| \quad \text{for any } \varepsilon \in (0, \delta_0^{1+\alpha}),\ t \in [0, T],\ \theta \in [\theta_1, \theta_2]. \tag{3.7}$$

Substituting (3.6) into (3.7) we obtain that

$$\rho(x_\varepsilon(t), x_0(t + [\theta_1, \theta_2])) \le \varepsilon^{1/(1+\alpha)} M_2 \left(1 + M_1 \varepsilon^{\alpha/(1+\alpha)}\right) \quad \text{for any } \varepsilon \in (0, \delta_0^{1+\alpha}),\ t \in [0, T]. \tag{3.8}$$

Assume now that (3.3) is not true, thus there exist $\delta_* > 0$ and sequences $\{\varepsilon_k\}_{k \in \mathbb{N}} \subset (0, \delta_0^{1+\alpha})$, $\varepsilon_k \to 0$ as $k \to \infty$, and $\{t_k\}_{k \in \mathbb{N}} \subset [0, T]$ such that

$$x_{\varepsilon_k}(t_k) \notin B_{\delta_*}(x_0(t_k + \Theta)) \quad \text{for any } k \in \mathbb{N}. \tag{3.9}$$

Without loss of generality we may assume that $\{x_k\}_{k \in \mathbb{N}}$ and $\{t_k\}_{k \in \mathbb{N}}$ are converging. From (3.8) we have that there exists $\theta_* \in [\theta_1, \theta_2]$ such that

$$x_k(t) \to x_0(t + \theta_*) \quad \text{as } k \to \infty \tag{3.10}$$

uniformly with respect to $t \in [0, T]$. By using [13] or ([12], Theorem p. 387) we can conclude from (3.10) that $f_{x_0}(\theta_*) = 0$. On the other hand, from (3.9) we have that $x_0(t_0 + \theta_*) \notin B_{\delta_*/2}(x_0(t_0 + \Theta))$, where $t_0 = \lim_{k \to \infty} t_k$, and thus $x_0(\theta_*) \notin B_{\delta_*/2}(x_0(\Theta))$. This contradiction proves (3.3) and thus the proof is complete. □

A topological degree approach to prove the existence of periodic solutions to some classes of autonomous perturbed systems can be found in [1] and [7].

Remark 3.4 From the proof of Theorem 3.3 we see that Theorem 2.1 also provides information about the rate of the convergence of T-periodic solutions of (1.1) to a limit cycle of (1.8). In fact, from (3.8) we have that the distance between the graph of the T-periodic solution x_ε and the limit cycle x_0 is of order $\varepsilon^{1/(1+\alpha)}$, where $\alpha > 0$ is any positive constant.

We are now in a position to establish some new existence results of T-periodic solutions to (1.1). First, by using Theorem 3.3 we state in Corollary 3.5 a generalization of the following Malkin's theorem, see ([13] and ([12], Theorems pp. 387 and 392), (the same result with a more rigorous proof is also given in ([11], Theorem 1)). In fact, in Corollary 3.5 the Malkin's regularity assumptions are weakened to conditions (1.2) and moreover $(f_{x_0})'(\theta_0)$ can be 0.

Malkin's theorem *Let $\psi \in C^3$, $\phi \in C^2$. Let x_0 be a nondegenerate T-periodic limit cycle of (1.8). Assume that there exists $\theta_0 \in [0,T]$ such that $f_{x_0}(\theta_0) = 0$ and*

$$(f_{x_0})'(\theta_0) \neq 0. \tag{3.11}$$

Then for all $\varepsilon > 0$ sufficiently small system (1.1) possesses a T-periodic solution x_ε satisfying

$$x_\varepsilon(t) \to x_0(t+\theta_0) \quad \text{as } \varepsilon \to 0, \quad t \in [0,T]. \tag{3.12}$$

Corollary 3.5 *Assume that ψ and ϕ satisfy conditions (1.2). Let x_0 be a nondegenerate T-periodic limit cycle of (1.8). Assume that there exists $\theta_0 \in [0,T]$ such that*

$$f_{x_0}(\theta_0) = 0 \quad and \quad f_{x_0} \ is\ strictly\ monotone\ at\ \theta_0. \tag{3.13}$$

Then for all $\varepsilon > 0$ sufficiently small system (1.1) possesses a T-periodic solution x_ε satisfying (3.12).

The proof of Corollary 3.5 is a direct consequence of Theorem 3.3 with $\theta_1 < \theta_0 < \theta_2$ sufficiently close to θ_0. We would like to observe that, under the regularity assumptions of the Malkin's theorem, the asymptotic stability of the resulting T-periodic solutions can be also established by means of the derivatives of the involved functions. Clearly, under the weaker regularity assumptions (1.2) this approach is impossible. On the other hand as shown in [17] some stability properties of the T-periodic solutions to (1.1) can be derived from the value of the degree $d(I - Q_\varepsilon, W_{V_\delta(\varepsilon)})$, where $V_\delta(\varepsilon)$ are the sets employed in the proof of Theorem 3.3.

The case when (3.11) is not satisfied was treated by Loud in [11], we show here that, by using Theorem 3.3, the conditions of a related Loud's existence result can be considerably simplified. Also for this case we do not provide here any result about the stability of the resulting periodic solutions as it has been done in [11]. In order to formulate the Loud's existence result we introduce some preliminary notations. First of all we need to translate and rotate the axes in such a way that $x_0(0) = 0$ and $\dot{x}_0(0) = \left([x_0(0)]^1, 0, ..., 0\right)$. Let $x(\cdot, \xi, \varepsilon)$ be the solution of (1.1) satisfying $x(0, \xi, \varepsilon) = \xi$. Let $F(\xi, \varepsilon) = x(T, \xi, \varepsilon) - \xi$, since the limit cycle x_0 is nondegenerate then $n-1$ equations of the system $F(\xi, \varepsilon) = 0$ can be solved near 0 with respect to some ξ^k, where $k \in \{1, 2, ..., n\}$ and as a result we obtain a scalar equation $H(u, \varepsilon) = 0$. Let D_{x_0} be the discriminant of the equation

$$\frac{1}{2}\frac{\partial^3 H}{\partial u^2 \partial \varepsilon}(0,0)m^2 + \frac{1}{2}\frac{\partial^3 H}{\partial u \partial \varepsilon^2}(0,0)m + \frac{1}{6}\frac{\partial^3 H}{\partial \varepsilon^3}(0,0) = 0.$$

We can now formulate the Loud's existence result, ([11], Theorem 2).

Loud's theorem. *Let $\psi \in C^3$, $\phi \in C^2$. Let x_0 be a nondegenerate T-periodic limit cycle of (1.8). Assume that for some $\theta_0 \in [0,T]$ satisfying $f_{x_0}(\theta_0) = 0$ we have $(f_{x_0})'(\theta_0) = 0$. Finally, assume that*

$$D_{x_0} > 0 \quad and \quad (f_{x_0})''(\theta_0) = 0. \tag{3.14}$$

Then, for all $\varepsilon > 0$ sufficiently small, system (1.1) has a T-periodic solution x_ε satisfying (3.12).

In the case when $f_{x_0}(\cdot)$ is identically zero Loud in [11] has derived from the above theorem an important result on the existence of T-periodic solutions to (1.1) near x_0. But even in the case when $(f_{x_0})'''(\theta_0) \neq 0$ to verify (3.14) is not a feasible problem (here it is assumed $\phi \in C^3$). This is the reason why it is of interest to state the following result which is a particular case of Corollary 3.5.

Corollary 3.6 *Let $\psi \in C^1$, $\phi \in C^3$. Let x_0 be a nondegenerate T-periodic limit cycle of (1.8). Assume that for some $\theta_0 \in [0,T]$ we have*

$$f_{x_0}(\theta_0) = f'_{x_0}(\theta_0) = f''_{x_0}(\theta_0) = 0, \quad f'''_{x_0}(\theta_0) \neq 0.$$

Then for all $\varepsilon > 0$ sufficiently small system (1.1) has a T-periodic solution x_ε satisfying (3.12).

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